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Programming**

Churlzu Lim
University of North Carolina at Charlotte, USA

Theodore S. Glickman
Department of Decision Sciences
The George Washington University, USA

Hanif D. Sherali
Virginia Polytechnic Institute and State University. USA

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Churlzu Lim*, Hanif D. Sherali†, Theodore S. Glickman‡

* University of North Carolina at Charlotte, corresponding author

† Virginia Polytechnic Institute and State University

‡ George Washington University

Abstract

In this paper, we consider a Cost of Quality (CoQ) optimization problem that finds an optimal allocation of prevention and inspection resources to minimize the expected total quality costs under a prevention-appraisal-failure framework, where the quality costs in the proposed model are involved with prevention, inspection, and correction of internal and external failures. Commencing with a simple structure of the problem, we progressively increase the complexity of the problem by accommodating realistic scenarios regarding preventive, appraisal, and corrective actions. The resulting problem is formulated as zero-one polynomial program, which can be solved either directly using a mixed-integer nonlinear programming (MINLP) solver such as BARON, or using a more conventional mixed-integer programming (MILP) solver such as CPLEX after performing an appropriate linearization step. We present a case study from the literature to illustrate how the proposed model can be utilized to find optimal inspection and prevention strategies, and to analyze sensitivity with respect to different cost parameters. We also provide a comparative numerical study of using the aforementioned solvers to optimize the respective model formulations. The results provide insights into the use of such quantitative methods for optimizing the cost of quality, and indicate the efficacy of using the linearized MILP

model for this purpose.

Keywords: Cost of Quality, zero-one Polynomial Programming, Linearization

1 Introduction

In the quality management literature, measuring and reporting the so-called *cost of quality* (CoQ) has been extensively studied as part of an effort to improve quality and reduce costs (see [17, 27, 30, 24] for surveys of CoQ). A widely used CoQ approach is the *prevention-appraisal-failure* (P-A-F) model of Feigenbaum [7], which was also adopted by the American Society for Quality [4]. The P-A-F model uses the costs of prevention, appraisal, and failure to describe the *total quality costs*. The failure cost can be further partitioned into two types: internal and external failures [14]. Prevention costs include the costs of activities that need to be performed in order to prevent poor quality in goods or services (e.g., quality planning, training, and process control). Appraisal costs are incurred when inspecting or evaluating goods or services to check if quality standards are met throughout the production processes. Both internal and external failures are caused by defective designs, materials, or processes. A failure that occurs before a good or service is delivered to the customer is referred to as an internal failure. When the internal failure is detected by an appraisal activity, correctional steps need to be performed, and thus, incur internal failure costs (e.g., scrap and rework). On the other hand, failures occurring when the good or service is used by or rendered to customers are external failures, which incur external failure costs (e.g., return, refund, complaint adjustment, and warranty charges).

Note that the trade-off between the quality improvement (i.e., prevention and appraisal) and the failure costs is evident because implementing more prevention and appraisal activities would reduce the likelihood of failures. In the prevailing conventional view on this trade-off, the prevention and

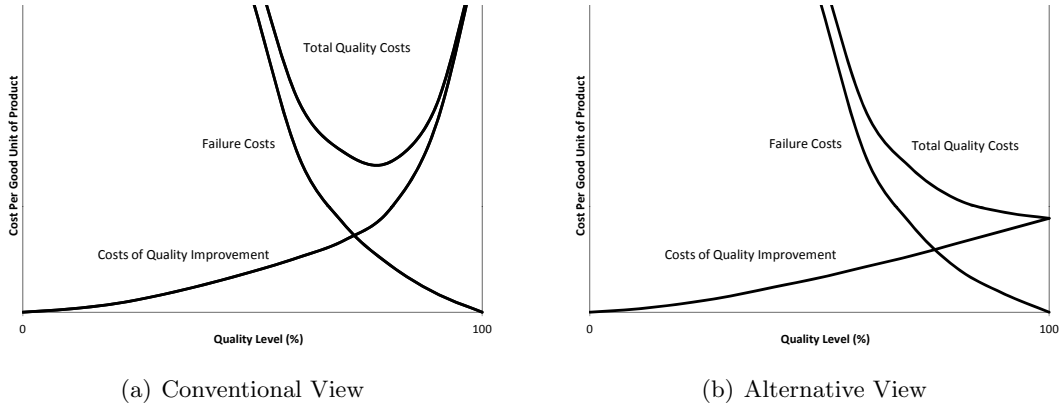


Figure 1: The conventional view suggests the existence of an economic equilibrium point, while the alternative view advocates that the minimum total quality cost is attained at the 100% quality level.

appraisal costs for achieving perfect quality become too large to justify, and hence, there exists an optimal quality level that minimizes total quality costs [14]. On the other hand, it has also been argued that zero defects can be attained at a finite cost, and the total cost is non-increasing in quality level. Therefore, it is always better to pursue perfect quality (i.e., 0% defective) if possible [25, 18] (see the illustration in Figure 1 for the conventional and alternative views). This alternative view is supported by the fact that quality improvement costs have decreased significantly due to advances in technology (e.g., machine processing that eliminates human errors). Nonetheless, the resolution between these two viewpoints depends on the specific process and the related cost parameters [5]. Furthermore, as indicated in [31], practice managers still face decision-making problems in order to determine how much to spend on quality improvement initiatives.

In this study, we consider an optimization problem that finds an optimal quality management plan to minimize the total quality cost. Specifically, in order to provide an optimal strategy that prescribes preventive and appraisal actions, we propose a zero-one polynomial programming model that minimizes the expected total quality cost under budgetary constraints. We also present extensions of the proposed basic model to accommodate additional uncertainties in the effectiveness of preventive and/or appraisal actions. Through a numerical example and sensitivity study, we provide insights into the effects of various cost parameters. We also provide a comparative study

of computational efficacy of two solution tools: CPLEX [11] and BARON [23].

The remainder of this paper is organized as follows. We first review the related literature briefly in Section 2. Our proposed mathematical programming model formulations with increasing degrees of features and complexity are discussed in Section 3. The resulting zero-one polynomial programming problems are linearized in Section 4 to derive equivalent linear zero-one mixed-integer programs (MIPs) by applying a suitable linearization strategy. We present an illustrative example along with insights on the sensitivity with respect to problem parameters in Section 5. Finally, a comparative numerical study is presented in Section 6, and we close with some concluding remarks in Section 7.

2 Literature Review

Conventional CoQ analyses address two types of costs: one incurred from the effort to improve quality, and the other incurred from any deficient quality of the resulting good or service. The former includes prevention and appraisal costs, while the latter includes internal and external failure costs [14]. These are also referred to as *conformance* and *non-conformance* costs, respectively [6]. The conformance costs are incurred while fulfilling the stated requirements of customers, whereas non-conformance costs are incurred due to failures of the product or service. Besides accounting for just rework and/or return costs, the non-conformance costs sometimes include opportunity and intangible costs (e.g., the cost of lost customers) as well [10, 28]. As the focus of this paper is on the optimization of CoQ, we do not delve into a detailed review of quality costing approaches, and we refer interested readers to several comprehensive survey papers such as [17, 27, 30, 24]. Accordingly, our review is mainly focused on optimization models and solution approaches in what follows.

The literature contains a few basic attempts to model and solve CoQ optimization problems. Pertaining to the conventional view of the trade-off between conformance and non-conformance

costs (see Figure 1(a)), Lundvall and Juran [14] presented three methods to find a suitable level of quality costs. The first method relies on market data, where the level of quality costs is determined to be the one that is used by other companies in the market. In the second method, companies determine the level of quality costs by budgeting inspection, prevention, and failure costs. However, since such budgeting is performed for individual elements of CoQ (i.e., inspection and prevention costs are determined without simultaneous consideration), the resulting solutions only sub-optimize the overall CoQ. In the third method, curves of conformance and non-conformance quality costs as depicted in Figure 1(a) are estimated, and accordingly, the optimal conformance quality level is determined. Among these three methods, the second one is closely related to the CoQ optimization model proposed in Section 3; however, unlike existing work in this area, we consider all the salient elements of CoQ simultaneously in order to determine an optimal plan for prevention and appraisal activities under budgetary restrictions via a mathematical programming model.

To our knowledge, only a few studies have applied mathematical programming approaches to CoQ optimization. Ramudhin et al. [21] formulated a general mixed-integer nonlinear program (GMINLP) for minimizing CoQ-embedded costs in a supply chain design problem. In their model, the objective function consists of costs involved with the procurement of raw material, transportation between supplier-plants and plant-customers, total quality at suppliers, and production (both with fixed and variable costs). The total quality cost at a supplier is represented as the CoQ at a given defective rate. This model was further extended to include the total quality at plants by Alzaman et al. [1], where a gradient search method was proposed for solving the GMINLP. Castillo-Villa et al. [5] presented an alternative zero-one nonlinear program for optimizing a supply chain design, where the objective function is the profit that is given by the expected revenue minus the CoQ at the plants, and minus the costs incurred for materials, production, and transportation. Instead of finding an exact solution, the authors employed a simulated annealing heuristic as the

solution method.

Yasin et al. [31] proposed a framework for applying optimal control theory to the second method of [14]. This framework was also employed in a building project that is focused on the impact of failures [9]. Considering only inspection (i.e., appraisal) strategies, Oppermann et al. [16] used dynamic programming to determine an inspection strategy for minimizing quality costs. There also exist some studies concerning the optimization of quality costs for construction project management [20, 15]. In this literature, the trade-off between cost and quality is modeled as a multi-objective optimization problem, which is then solved heuristically. The multi-objective model in [20] considers three objectives: construction cost, time, and quality level. It is referred to as the *discrete time-cost trade-off problem* (DTCTP) and is solved using a particle swarm optimization procedure. Alternatively, [19] formulates DTCTP as a mixed-integer program that optimizes one of the objective functions while constraining the other two objectives within desired bounds, and [15] considers minimizing the construction cost and maximizing the system reliability, and solves this multi-objective program using an ant colony optimization technique.

3 Model Formulation

In this section, we present a model that accommodates the four costs (prevention, appraisal, internal, and external costs) within a decision-making framework. Toward this end, consider a project, a manufacturing process, or a supply chain that involves a set of interrelated activities with given activity times and precedence relationships. In such a context, suppose further that a quality inspection (i.e., an appraisal) is performed at one or more checkpoints to determine whether the work product at that stage is defective (i.e., an internal failure has occurred), in which case certain corrective steps are undertaken, such as rework or replacement. On the other hand, the occurrence of an internal failure that goes undetected could pose a problem later on, along with an accompa-

nying cost, after all the activities have been completed, i.e., a related external failure could occur. Finally, suppose that the risk of certain internal failures can be eliminated (i.e., prevented) by taking certain precautions, such as training or quality planning, before any of the related activities are performed. Note that this framework does not account for any internal or external failures that might occur in the absence of defects, e.g., as a result of unrecognized design deficiencies that do not appear as defects.

The following assumptions apply to the different models that we develop below with progressively increasing complexity for minimizing the cost of quality:

1. In combination, the inspections at the various checkpoints cover the entire set of internal failures that can possibly occur in the course of performing the required activities.
2. The inspection performed at each checkpoint can result in the recognition of a variety of different internal failures with different corrective requirements, or in the finding that there are no internal failures.
3. At each checkpoint, we can estimate the cost of inspection, the probabilities of the various possible internal failure outcomes, and the associated costs of correcting these failures. We can also estimate the costs of the various possible preventive actions and external failures.

Basic Model

In the basic model that we present first, the objective is to identify a set of preventive actions that minimizes the total cost of prevention plus the expected total cost of correcting internal failures, given a predetermined set of inspection checkpoints and subject to specified limits on these two costs. To formulate this model, we introduce the following notation:

Sets and parameters:

I = the set of all possible internal failures, indexed by i .

J = the set of all checkpoints j , arranged in order of inspections performed.

$J(i)$ = the subset of checkpoints j at which internal failure i can be detected.

p_i = the probability that internal failure i occurs.

q_{ij} = the conditional probability of detecting internal failure i at checkpoint $j \in J(i)$, given that it has occurred.

$q'_{ij} = q_{ij} \prod_{n < j, n \in J(i)} (1 - q_{in})$ = the conditional probability that internal failure i was undetected at any checkpoint prior to $j \in J(i)$ and is detected at checkpoint j , given that the internal failure i has occurred.

c_i^P = the cost of preventing internal failure i .

c_{ij}^{IF} = the cost of correcting internal failure i if detected at checkpoint $j \in J(i)$.

b^P = an upper limit on the total cost of preventive actions.

b^{IF} = an upper limit on the expected total cost of correcting internal failures.

Decision variables:

$x_i = 1$ if internal failure i is prevented, and $x_i = 0$ otherwise.

Assuming that the internal failures are independent, the total cost of prevention plus the expected total cost of internal failure correction is given by

$$\sum_{i \in I} c_i^P x_i + \sum_{i \in I} \sum_{j \in J(i)} c_{ij}^{IF} q'_{ij} p_i (1 - x_i).$$

The model for minimizing this cost subject to separate budget restrictions for prevention and internal failure correction is then formulated as follows:

$$\mathbf{P1:} \quad \text{Minimize} \quad c_i^P x_i + \sum_{i \in I} \sum_{j \in J(i)} c_{ij}^{IF} q'_{ij} p_i (1 - x_i) \quad (1a)$$

$$\text{subject to} \quad \sum_{i \in I} c_i^P x_i \leq b^P \quad (1b)$$

$$\sum_{i \in I} \sum_{j \in J(i)} c_{ij}^{IF} q'_{ij} p_i (1 - x_i) \leq b^{IF} \quad (1c)$$

$$x_i \in \{0, 1\} \text{ for all } i \in I, \quad (1d)$$

where (1b) and (1c) are the budget constraints for prevention and internal failure costs, respectively, and (1d) represents the logical restrictions.

Extension I: Accommodating external failures

Suppose that an external failure can occur if and only if a related internal failure has occurred but was not detected at any checkpoint (in reality, some external failures may be triggered by a combination of internal failures rather than a single one). To incorporate the expected cost of such external failures in the model, we introduce the following additional notation:

K = the set of all possible external failures, indexed by k .

r_{ik} = the conditional probability of external failure k occurring, given that internal failure i has occurred.

$r'_{ik} = r_{ik} \prod_{j \in J(i)} (1 - q_{ij})$ = the conditional probability of internal failure i being undetected at any checkpoint and external failure k occurring, given that internal failure i has occurred.

c_k^{EF} = the cost of external failure k .

b^{EF} = an upper limit on the expected total cost of external failures.

The extended model that also accounts for the expected total cost of external failures along with the associated budgetary restriction is then formulated as follows:

$$\mathbf{P2:} \quad \text{Minimize} \quad c_i^P x_i + \sum_{i \in I} \sum_{j \in J(i)} c_{ij}^{IF} q'_{ij} p_i (1 - x_i) + \sum_{i \in I} \sum_{k \in K} c_k^{EF} r'_{ik} p_i (1 - x_i) \quad (2a)$$

$$\text{subject to} \quad \sum_{i \in I} c_i^P x_i \leq b^P \quad (2b)$$

$$\sum_{i \in I} \sum_{j \in J(i)} c_{ij}^{IF} q'_{ij} p_i (1 - x_i) \leq b^{IF} \quad (2c)$$

$$\sum_{i \in I} \sum_{k \in K} c_k^{EF} r'_{ik} p_i (1 - x_i) \leq b^{EF} \quad (2d)$$

$$x_i \in \{0, 1\} \text{ for all } i \in I. \quad (2e)$$

Extension II: Accommodating selection of checkpoints

When a number of candidate checkpoints exist but no decision has been made as to which ones to use, we introduce the following additional factors:

c_j^A = the cost of performing an inspection at checkpoint j .

b^A = an upper limit on the total cost of the inspections.

$y_j = 1$ if candidate checkpoint j is selected, and $y_j = 0$ otherwise. This is a **decision variable**,

which is also denoted in vector form by $\mathbf{y} = (y_1, y_2, \dots, y_{|J|})$. ($|\cdot|$ denotes the cardinality of \cdot).

$q''_{ij}(\mathbf{y}) = q_{ij} \prod_{n < j, n \in J(i)} (1 - q_{in} y_n)$ = the conditional probability that internal failure i was undetected at any available checkpoint prior to $j \in J(i)$ and is detected at an established checkpoint j , given that the internal failure i has occurred.

$r''_{ik}(\mathbf{y}) = r_{ik} \prod_{j \in J(i)} (1 - q_{ij} y_j)$ = the conditional probability of internal failure i being undetected at any available checkpoint and external failure k occurring, given that internal failure i has occurred.

When these additional factors are incorporated within the model, the resulting formulation for minimizing all the previous costs plus the total cost of inspection along with its associated budgetary restriction is then given as follows:

$$\mathbf{P3:} \quad \text{Minimize} \quad \sum_{i \in I} c_i^P x_i + \sum_{i \in I} \sum_{j \in J(i)} c_{ij}^{IF} q''_{ij}(\mathbf{y}) p_i (1 - x_i) y_j$$

$$+ \sum_{i \in I} \sum_{k \in K} c_k^{EF} r_{ik}''(\mathbf{y}) p_i (1 - x_i) + \sum_{j \in J} c_j^A y_j \quad (3a)$$

$$\text{subject to } \sum_{i \in I} c_i^P x_i \leq b^P \quad (3b)$$

$$\sum_{i \in I} \sum_{j \in J(i)} c_{ij}^{IF} q_{ij}''(\mathbf{y}) p_i (1 - x_i) y_j \leq b^{IF} \quad (3c)$$

$$\sum_{i \in I} \sum_{k \in K} c_k^{EF} r_{ik}''(\mathbf{y}) p_i (1 - x_i) \leq b^{EF} \quad (3d)$$

$$\sum_{j \in J} c_j^A y_j \leq b^A \quad (3e)$$

$$x_i \in \{0, 1\} \text{ for all } i \in I \quad (3f)$$

$$y_j \in \{0, 1\} \text{ for all } j \in J. \quad (3g)$$

Extension III: Accommodating imperfect preventive actions

We can expand the foregoing model to allow for the possibility that preventive actions may not be 100% effective, in which case we need to introduce the following additional probabilities that reflect the effectiveness of preventive actions:

f_i = the probability of definitively preventing internal failure i .

Accordingly, we replace each $p_i(1 - x_i)$ with $p_i(1 - f_i x_i)$, where $0 < f_i \leq 1$, within Problem P3 as follows:

$$\begin{aligned} \mathbf{P4:} \quad \text{Minimize} \quad & \sum_{i \in I} c_i^P x_i + \sum_{i \in I} \sum_{j \in J(i)} c_{ij}^{IF} q_{ij}''(\mathbf{y}) p_i (1 - f_i x_i) y_j \\ & + \sum_{i \in I} \sum_{k \in K} c_k^{EF} r_{ik}''(\mathbf{y}) p_i (1 - f_i x_i) + \sum_{j \in J} c_j^A y_j \end{aligned} \quad (4a)$$

$$\text{subject to } \sum_{i \in I} c_i^P x_i \leq b^P \quad (4b)$$

$$\sum_{i \in I} \sum_{j \in J(i)} c_{ij}^{IF} q_{ij}''(\mathbf{y}) p_i (1 - f_i x_i) y_j \leq b^{IF} \quad (4c)$$

$$\sum_{i \in I} \sum_{k \in K} c_k^{EF} r_{ik}''(\mathbf{y}) p_i (1 - f_i x_i) \leq b^{EF} \quad (4d)$$

$$\sum_{j \in J} c_j^A y_j \leq b^A \quad (4e)$$

$$x_i \in \{0, 1\} \text{ for all } i \in I \quad (4f)$$

$$y_j \in \{0, 1\} \text{ for all } j \in J. \quad (4g)$$

Extension IV: Accommodating imperfect corrective actions

We can further extend the model to account for the possibility that corrective actions such as rework are not always successful the first time and may have to be repeated, incurring additional costs. In other words, an internal failure may be detected at a checkpoint and corrected, but then the same internal failure might recur and be detected again at a subsequent checkpoint. Assume that the conditional detection probability q_{ij} remains the same. To accommodate the uncertainty of corrective actions, we introduce the following factor:

\bar{p}_i = the conditional probability of internal failure i recurring, given that the internal failure i has occurred and corrective action was made at an earlier checkpoint.

Then, the probability that internal failure i is detected at an established checkpoint j becomes

$$\begin{aligned} & Pr[i \text{ is detected at } j] \\ &= Pr[i \text{ is detected at } j \text{ and } i \text{ has occurred}] \\ &= p_i Pr[i \text{ is detected at } j | i \text{ has occurred}] \\ &= p_i Pr[i \text{ is detected at } j \text{ and } i \text{ has been detected before } j | i \text{ has occurred}] \end{aligned}$$

$$\begin{aligned}
& + p_i Pr[i \text{ is detected at } j \text{ and } i \text{ has been undetected before } j | i \text{ has occurred}] \\
& = p_i \left(1 - \prod_{n < j, n \in J(i)} (1 - q_{in} y_n) \right) \\
& \quad \times Pr[i \text{ is detected at } j | i \text{ has occurred and has been detected before } j] \\
& + p_i \left(\prod_{n < j, n \in J(i)} (1 - q_{in} y_n) \right) \\
& \quad \times Pr[i \text{ detected at } j | i \text{ has occurred and has been undetected before } j] \\
& = p_i \bar{p}_i q_{ij} \left(1 - \prod_{n < j, n \in J(i)} (1 - q_{in} y_n) \right) + p_i q_{ij} \prod_{n < j, n \in J(i)} (1 - q_{in} y_n) \\
& = p_i [\bar{p}_i q_{ij} + (1 - \bar{p}_i) q''_{ij}(\mathbf{y})].
\end{aligned}$$

Similarly, the probability of external failure k occurring is

$$\begin{aligned}
& Pr[\text{external failure } k] \\
& = p_i Pr[\text{external failure } k | i \text{ has occurred}] \\
& = p_i Pr[\text{external failure } k \text{ and } i \text{ was detected} | i \text{ has occurred}] \\
& \quad + p_i Pr[\text{external failure } k \text{ and } i \text{ was undetected} | i \text{ has occurred}] \\
& = p_i Pr[\text{external failure } k | i \text{ has occurred and was detected}] Pr[i \text{ was detected} | i \text{ has occurred}] \\
& \quad + p_i Pr[\text{external failure } k | i \text{ has occurred and was undetected}] Pr[i \text{ was undetected} | i \text{ has occurred}] \\
& = p_i \bar{p}_i r_{ik} \left(1 - \prod_{j \in J(i)} (1 - q_{ij} y_j) \right) + p_i r_{ik} \prod_{j \in J(i)} (1 - q_{ij} y_j) \\
& = p_i (\bar{p}_i r_{ik} + (1 - \bar{p}_i) r''_{ik}(\mathbf{y})).
\end{aligned}$$

Therefore, the model formulation with uncertain corrective actions is given as follows:

$$\begin{aligned} \mathbf{P5:} \quad \text{Minimize} \quad & \sum_{i \in I} c_i^P x_i + \sum_{i \in I} \sum_{j \in J(i)} c_{ij}^{IF} p_i [\bar{p}_i q_{ij} + (1 - \bar{p}_i) q_{ij}''(\mathbf{y})] (1 - f_i x_i) y_j \\ & + \sum_{i \in I} \sum_{k \in K} c_k^{EF} p_i [\bar{p}_i r_{ik} + (1 - \bar{p}_i) r_{ik}''(\mathbf{y})] (1 - f_i x_i) + \sum_{j \in J} c_j^A y_j \end{aligned} \quad (5a)$$

$$\text{subject to} \quad \sum_{i \in I} c_i^P x_i \leq b^P \quad (5b)$$

$$\sum_{i \in I} \sum_{j \in J(i)} c_{ij}^{IF} p_i [\bar{p}_i q_{ij} + (1 - \bar{p}_i) q_{ij}''(\mathbf{y})] (1 - f_i x_i) y_j \leq b^{IF} \quad (5c)$$

$$\sum_{i \in I} \sum_{k \in K} c_k^{EF} p_i [\bar{p}_i r_{ik} + (1 - \bar{p}_i) r_{ik}''(\mathbf{y})] (1 - f_i x_i) \leq b^{EF} \quad (5d)$$

$$\sum_{j \in J} c_j^A y_j \leq b^A \quad (5e)$$

$$x_i \in \{0, 1\} \quad \text{for all } i \in I \quad (5f)$$

$$y_j \in \{0, 1\} \quad \text{for all } j \in J. \quad (5g)$$

Note that P5 reduces to P4 if $\bar{p}_i = 0$, and that P3, P4, and P5 are zero-one nonlinear programs having a similar structure. In the next section, we present a linearization strategy for P3, which can also be applied to P4 and P5 with straightforward modifications.

4 Handling Nonlinearity

The proposed formulations of the CoQ optimization problem include nonlinear products of zero-one variables. In this section, we discuss how to handle such nonlinearity to facilitate solving the problem. Specifically, consider P3 that has the following products of binary variables in eqref:ext2-a:

$$(1 - x_i) y_j \prod_{n < j, n \in J(i)} (1 - q_{in} y_n) \quad (6a)$$

and

$$(1 - x_i) \prod_{j \in J(i)} (1 - q_{ij} y_j). \quad (6b)$$

The nonlinear products in (6) can be linearized by introducing additional variables and constraints.

In light of the simple structure of nonlinear terms having only zero-one variables, we consider a straightforward application of a standard linearization procedure [8]. For the sake of simplicity in presentation, we substitute $\bar{x}_i \equiv 1 - x_i$ in the subsequent formulations. Furthermore, for $i \in I$ and $j \in J$, let $J(i, j) = \{n \in J(i) : n < j\}$. Then, we have

$$\prod_{n < j, n \in J(i)} (1 - q_{in} y_n) = \sum_{S \subseteq J(i, j)} (-1)^{|S|} \prod_{n \in S} q_{in} y_n = \sum_{S \subseteq J(i, j)} (-1)^{|S|} \left(\prod_{n \in S} q_{in} \right) \left(\prod_{n \in S} y_n \right) \quad (7a)$$

and

$$\prod_{j \in J(i)} (1 - q_{ij} y_j) = \sum_{S \subseteq J(i)} (-1)^{|S|} \left(\prod_{j \in S} q_{ij} \right) \left(\prod_{j \in S} y_j \right). \quad (7b)$$

Therefore, denoting $q_S^i \equiv (-1)^{|S|} \prod_{n \in S} q_{in}$ for any $S \subseteq J$, (6) can be rewritten as follows:

$$\sum_{S \subseteq J(i, j)} q_S^i \left(\bar{x}_i y_j \prod_{n \in S} y_n \right) \quad (8a)$$

and

$$\sum_{S \subseteq J(i)} q_S^i \left(\bar{x}_i \prod_{j \in S} y_j \right). \quad (8b)$$

For ease in presentation, we denote the decision variables \bar{x}_i and y_j using a generic notation z_m for $m \in M \equiv \{1, 2, \dots, |I|, |I| + 1, \dots, |I| + |J|\}$. Accordingly, each product of variables that appear

in (8) is represented as

$$\prod_{m \in L} z_m, \tag{9}$$

where $L \subseteq M$ is the set of appropriate indices that correspond to the variables constituting the particular product term.

Following the standard linearization approach of Glover and Woolsey [8], we substitute each product of variables $\prod_{m \in L} z_m$ by a single variable $v_L \geq 0$, and add the following constraints to the linearized formulation:

$$v_L \leq z_m, \quad \forall m \in L \tag{10a}$$

$$\sum_{m \in L} z_m - v_L \leq |L| - 1. \tag{10b}$$

Note that, if there exists an $m \in L$ such that $z_m = 0$, then we have $v_L = 0$ from (10a), whereas if $z_m = 1$ for all $m \in L$, then we have $v_L = 1$ from (10b).

Remark: Whereas mixed-integer linear programming (MILP) solvers can be used to optimize the equivalent linearized zero-one MIP versions of the CoQ optimization problem, there exist solvers (such as BARON [23], Couenne [3], LindoGlobal [13], and α BB [2]) that can directly solve the original mixed-integer nonlinear programming (MINLP) formulations. BARON is one such commonly used off-the-shelf solver, which utilizes a polyhedral branch-and-cut method [29]. In the next section, we will compare the computational performance of BARON for solving the MINLP formulations with that of CPLEX, a widely-used commercial MIP solver [11], for optimizing the corresponding equivalent, linearized MILP formulations.

5 Illustrative Example

To illustrate the proposed CoQ optimization formulations (P3, in particular, for this case) along with the associated linearization strategy, we adapt an Order Entry Department (OED) problem faced by a manufacturing company as presented in Kalagnanam and Matsumura [12]. The company, called PSI Inc., manufactures high-technology measurement instruments, which can be customized to accommodate configuration changes and add-ons in response to customer needs. In the case study, they consider a quality cost analysis in OED, which performs two main functions: preparing quotations for potential customers and taking actual sales orders. Whereas price quotes are requested by the sales representatives of the company, sales orders are directly placed by customers who have obtained quotes earlier. Efficacy in the order entry process can greatly influence the overall quality of the operations in the company because errors made during the quotation and order processes incur significant costs in the downstream manufacturing process.

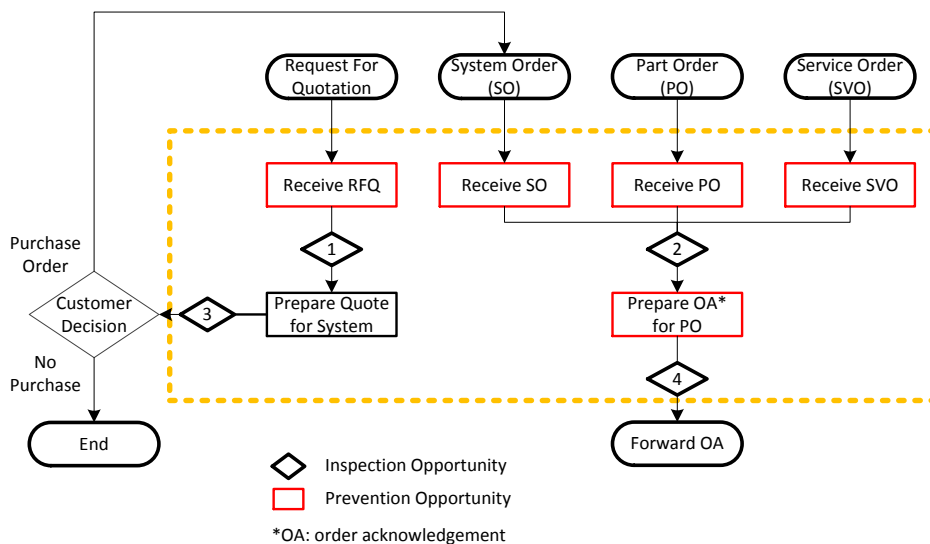


Figure 2: Overview of the work flow of the Order Entry Department: Potential inspection points and prevention opportunities are indicated.

Using the case study as a backbone, we modify the scope of the case to establish a CoQ optimization problem that encompasses inspection points and prevention opportunities in the Order

Entry Department, the Sales Department (SD), and the Service Department (SVD). (The OED work process is illustrated in Figure 2.) Based on the description in [12], we consider five particular types of internal failures (IF-1, IF-2, IF-3, IF-4, IF-5), each of which can cause one of four types of external failures (EF-1, EF-2, EF-3, EF-4). The internal failures IF-1, IF-2, and IF-3 include errors made in the Order Acknowledgement (OA) procedure when it was prepared. IF-1 consists of errors made in OA that need only *Rework* to correct the error when it is detected after the OA was transferred to another department (hence, external failure EF-1). IF-2 contains errors in OA that need both *Rework* and *Change Order* to remedy the corresponding external failure EF-2. IF-3 represents errors in invoice-related data, which can cause payment delay by the final customer (EF-3). IF-4 and IF-5 are errors made by SD and SVD representatives, respectively, when they receive requests for quotations and service orders. Both internal failures can cause a return of the product (EF-4), which creates additional work for OED, which must then prepare a *Return Authorization*.

Internal Failure (i)	p_i	Prevention Cost (c_i^P)	External Failure (k)	c_k^{EF}	r_{ik}	Inspection Points (j)	q_{ij}	c_{ij}^{IF}
IF-1	0.7	10	EF-1	9.94	0.9	2	0.5	1.18
						4	0.8	1.39
IF-2	0.55	15	EF-2	16.82	0.9	2	0.4	1.73
						4	0.7	2.03
IF-3	0.2	3	EF-3	3.21	0.5	2	0.8	0.69
						4	0.6	0.81
IF-4	0.58	5	EF-4	3.22	0.9	1	0.8	0.96
						2	0.7	1.13
						3	0.6	1.33
						4	0.5	1.57
IF-5	0.58	5	EF-4	3.22	0.9	2	0.8	1.33
						4	0.6	1.57

Figure 3: Specification of parameters for the illustrative example. (Inspection costs at $j = 1, 2, 3,$ and 4 are $3, 3, 5,$ and $10,$ respectively.)

We consider four potential inspection points for detecting internal failures (see Figure 2). From

the estimated annual failure costs presented [12] in conjunction with human error rates of [26], we inferred individual costs and other parameter values under assumed failure and detection probabilities as summarized in Figure 3. Inspection costs at the four locations were set as $c_1^A = 3$, $c_2^A = 3$, $c_3^A = 5$, and $c_4^A = 10$, respectively. The problem was formulated as P3 and solved using CPLEX 12.4 [11]. Since the size of the problem was relatively small, the solver was able to generate an optimal solution instantly. The optimal solution is to select the inspection point at location 2, which is just before the OA preparation (see Figure 2 for the location of the inspection point). The optimal cost is 14.0112, which is expressed as a percentage of OED's annual salary and fringe benefits.

In order to examine the sensitivity of the solution with respect to the cost parameters, we experimented with increased and decreased values of a single cost parameter, while keeping the remaining costs the same as before. We identified ranges over which the optimal solution remains the same by sequentially increasing and decreasing the cost values, and the results are reported in Figures 4–6.

Internal Failure i	Current Prevention Cost	Current Solution	Range of c_i^P for which solution remains same	New Solution at Changing Point*
1	10	$x = (0, 0, 0, 0, 0)$ $y = (0, 1, 0, 0)$	$(3, \infty)$	$x = (1, 0, 0, 0, 0)$ $y = (0, 1, 0, 0)$
2	15		$(5, \infty)$	$x = (0, 1, 0, 0, 0)$ $y = (0, 1, 0, 0)$
3	3		$(0.1, \infty)$	$x = (0, 0, 1, 0, 0)$ $y = (0, 1, 0, 0)$
4	5		$(0.9, \infty)$	$x = (0, 0, 0, 1, 0)$ $y = (0, 1, 0, 0)$
5	5		$(0.9, \infty)$	$x = (0, 0, 0, 0, 1)$ $y = (0, 1, 0, 0)$

* Bounding values other than 0 and ∞ are changing points.

Figure 4: Sensitivity to prevention costs.

Note that no prevention is made in the current solution as we have $x_1 = x_2 = x_3 = x_4 = x_5 = 0$. Hence, it is intuitive to retain the same solution as prevention costs increase (see upper bounds of

ranges for c_i^P in Figure 4). On the other hand, as we decreased each prevention cost, we were able to identify a changing point where a preventive step for the corresponding internal failure is taken. For example, the prevention of internal failure 1 became optimal when the prevention cost was decreased to 3. Since a single prevention does not affect the occurrence of other internal failures, the remaining optimal values remain the same as in the current solution.

Inspection Point j	Current Inspection Cost	Current Solution	Range of c_j^A for which solution remains same	New Solution at Changing Point*
1	3	$x = (0, 0, 0, 0, 0)$ $y = (0, 1, 0, 0)$	$(0.3, \infty)$	$x = (0, 0, 0, 0, 0)$ $y = (1, 1, 0, 0)$
2	3		$[0, 7.1)$	$x = (0, 0, 0, 0, 0)$ $y = (0, 0, 0, 1)$
3	5		$(0.1, \infty)$	$x = (0, 0, 0, 0, 0)$ $y = (0, 1, 1, 0)$
4	10		$(5.9, \infty)$	$x = (0, 0, 0, 0, 0)$ $y = (0, 0, 0, 1)$

* Bounding values other than 0 and ∞ are changing points.

Figure 5: Sensitivity to inspection costs.

Unlike prevention efforts, conducting inspections may affect the detection probabilities of multiple internal failures, and in turn, the internal and external failure costs. Hence, the optimal solution at a changing point for inspection costs may have more than one value that differs from the corresponding value in the solution to the original problem. For example, as we increase the inspection cost at point 2 to 7.1, inspection point 2 is no longer optimal, and implementing inspection point 4 becomes optimal instead. That is, the new solution differs from the original solution in two places, y_2 and y_4 (see Figure 5). Also, we remark that an increase in the inspection cost for inspection point 2 resulted in a new solution, while decreasing the inspection costs at other points rendered alternative solutions. This is because the current solution is $y = (0, 1, 0, 0)$, i.e., only inspection point 2 is activated while others are not.

As far as the external failure cost changes are concerned, the solution was changed only when the external failure costs were increased. Similar to the case of inspection costs, new solutions do not

display any specific pattern. For example, when the cost of external failure 1 was increased to 30.5, prevention of internal failure 1 became optimal. However, increasing the cost of external failure 4 resulted in the prevention of internal failure 4 and an additional inspection at point 1 became optimal.

External Failure k	Current EF Cost	Current Solution	Range of c_j^A for which solution remains same	New Solution at Changing Point*
1	9.94	$x = (0, 0, 0, 0, 0)$ $y = (0, 1, 0, 0)$	[0, 30.5)	$x = (1, 0, 0, 0, 0)$ $y = (0, 1, 0, 0)$
2	16.82		[0, 44.1)	$x = (0, 0, 0, 0, 0)$ $y = (0, 0, 0, 1)$
3	3.21		[0, 144.5)	$x = (0, 0, 1, 0, 0)$ $y = (0, 1, 0, 0)$
4	3.22		[0, 24.6)	$x = (0, 0, 0, 0, 1)$ $y = (1, 1, 0, 0)$

* Bounding values other than 0 and ∞ are changing points.

Figure 6: Sensitivity to external failure costs.

6 Numerical Study

In order to examine the efficacy of implementing CPLEX to solve the linearized MILP formulation as opposed to using BARON for solving the original MINLP formulation as discussed in Section 4, we conducted a comparative study on a set of randomly generated test instances. For deriving test instances, we assume that the final product is composed of 2^m parts, and pairs of parts are assembled to form 2^{m-1} subassemblies. Subsequently, each pair of subassemblies is combined to make a larger subassembly, the number of which is reduced to half. Therefore, after assembling m layers for a total of $2^m - 1$ times, the final product is produced. Suppose that each of the 2^m parts can be defective (i.e., there are 2^m types of internal failures). Furthermore, assume that each part or subassembly (including the final product) can be inspected. Hence, there are $|J| = 2^{m+1} - 1$ potential inspection points (see Figure 7). Note that each internal failure can be inspected at no more than $(m+1)$ inspection points. We also assume that internal and external failures are mapped

one-to-one, i.e., $|I| = |K| = 2^m$.

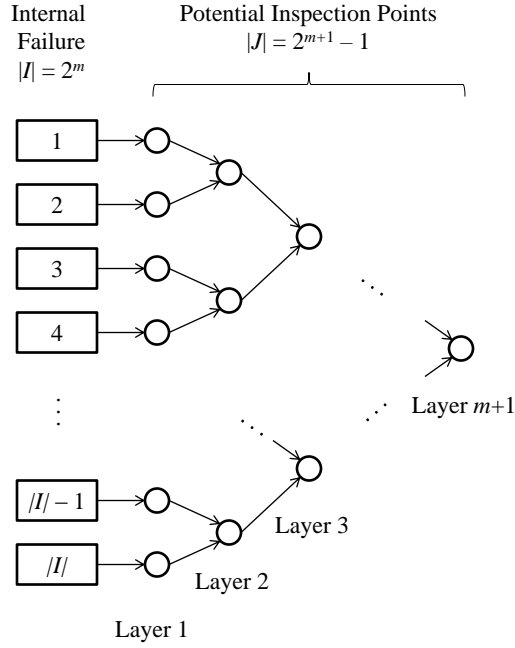


Figure 7: Specification of parameters for the comparative study.

For each internal failure, the probability of occurrence is generated from a uniform distribution defined on $(0.01, 0.3)$. Similarly, c_i^P (prevention costs) and c_k^{EF} (external failure costs) are sampled from uniform distributions defined on the intervals $(20, 50)$ and $(10, 30)$, respectively. The conditional probability of external failure k given that internal failure i has occurred, r_{ik} , is randomly generated on the interval $(0.8, 1.0)$.

To generate the correction costs of internal failures, it is assumed that early detection of an internal failure incurs less correction cost than late detection does. We also assume that the detection probability decreases as the inspection point is located close to the final product. Given m , if the inspection point is in layer L , then c_{ij}^{IF} and q_{ij} are randomly generated from $\left(\frac{10(L-1)}{m+1}, \frac{10L}{m+1}\right)$ and $\left(0.5 + \frac{0.5(L-1)}{m+1}, 0.5 + \frac{0.5L}{m+1}\right)$, respectively. Finally, the budget for each type of cost is drawn on an interval between half and three quarters of the maximum possible cost. For example, the budget for prevention cost is randomly selected on the interval $(0.5 \sum_{i=1}^n c_i^P, 0.75 \sum_{i=1}^n c_i^P)$.

Five sizes with $m = 3, 4, 5, 6, 7$ were considered for experimentation in this study, where 10

instances were generated for each size. C++ programs along with CPLEX 12.4 Concert Technology were used to formulate and solve the MILPs as in Section 4. On the other hand, the direct MINLP formulation was coded in GAMS (General Algebraic Modeling System) [22], and solved using the BARON solver [23]. For a fair comparison of the computational performances, we used the default values for all options when calling both CPLEX and BARON solvers, except for the time limit that was set to 3600 seconds to avoid excessively long solution times. Furthermore, all runs were conducted on a single computer equipped with a 2.53 GHz CPU, 4 GB memory, and the 32-bit Windows 7 operating system.

The results of the experimentation display that the performance of solving the MILP formulations using CPLEX dominates that of solving the original MINLP formulations using BARON, especially when the problem size is large. In Table 1, we report the average solution times in seconds for $m = 3, 4, 5, 6$, and 7. Note that, for $m = 3$, the average solution time for BARON (0.15) was slightly better than that for CPLEX (0.16). However, for $m = 4, 5$, and 6, CPLEX consumed significantly less solution times, ranging from 10.5% to 42.5% of those consumed by BARON on average. For $m = 7$, CPLEX was able to produce optimal solutions, whereas BARON provided optimal solutions only for two test instances within the time limit. The solution times for these two test instances were 1736.77 and 1778.93 seconds.

In addition to the average solution times, we also report the variations of solution times, which were measured as sample standard deviations, as well as the maximum and the minimum solution times of 10 test instances for each problem size. It is noteworthy that the solution times for CPLEX display larger variations than those for BARON. For example, for $m = 6$, the CPLEX runs have a sample standard deviation of 5.09, which is larger than the sample standard deviation (3.00) for the runs using BARON. Also, the CPLEX solution times varied from 3.5 to 20.0 with a range of 16.5, while this range was 10.6 for the solution times using BARON. Even considering the fact that

Size (m)	CPLEX Runs				BARON Runs			
	Average	SSD	Max	Min	Average	SDD	Max	Min
3	0.16	0.07	0.28	0.08	0.15	0.06	0.31	0.09
4	0.18	0.06	0.25	0.09	0.41	0.04	0.47	0.34
5	0.61	0.27	1.14	0.41	5.78	0.26	6.19	5.37
6	12.83	5.09	20.03	3.53	89.68	3.00	94.52	83.96
7	854.40	363.05	1347.88	436.99	> 3231.57	-	> 3600	1736.77

SSD: sample standard deviation

Table 1: Comparison of solution times of CPLEX and BARON runs in seconds: for $m = 7$, BARON runs were terminated by the time limit of 3600 seconds for 8 problems out of 10 test instances.

the average solution time for BARON was about seven times higher than that for CPLEX, the solution times for BARON display considerably lower variability.

Even though BARON terminated at the time limit of 3600 seconds when solving eight of the ten test problems having $m = 7$, we observed that the best objective values (i.e., upper bounds on the respective optimal values) provided by BARON at the end of the runs coincided with the optimal objective values obtained by CPLEX. However, the lower bounds generated by BARON using the branch-and-cut method [29] were not tight enough to enable an efficient termination of the runs based on the optimality gap, which is defined as $\frac{\text{upper bound} - \text{lower bound}}{\text{lower bound}} 100\%$. The average optimality gaps for these eight problems at termination was 65.9%.

7 Conclusion

In this paper, we consider a Cost of Quality (CoQ) optimization problem that finds an optimal portfolio of prevention and inspection decisions in order to minimize the expected total quality costs under a prevention-appraisal-failure (P-A-F) framework. In the proposed model, we incorporated four types of costs: prevention, inspection, internal, and external failure costs. The CoQ optimization problem was formulated as a zero-one polynomial program, which can be solved using conventional optimization methods (such as MINLP or MILP) after applying an appropriate linearization technique.

We adapted a case study from the literature [12] to illustrate how the proposed CoQ model can

be utilized. To provide further insights, we then conducted a sensitivity analysis of the model solution to various cost factors. We also investigated the numerical performance of two commercial solvers: CPLEX [11] and BARON [23]. CPLEX is widely used for solving linear and mixed-integer linear programs, while BARON is specialized to solve mixed-integer nonlinear programs using the branch-and-cut method [29]. Based on the numerical experimentation conducted on randomly generated test instances, solving the MILP formulation using CPLEX after linearization consumed about one-seventh of the time required to solve the original MINLP formulation using BARON. The computational efficiency of the former method became more pronounced as the problem size increased. Therefore, we advocate applying the linearization step to the original problem and solving the resulting MILP formulation for the proposed CoQ optimization problem.

To our knowledge, this is the first study that proposes a mathematical programming model for optimizing CoQ under a general setting of the P-A-F framework. With effective solution methods, the proposed model provides not only the optimal quality management strategy, but also insights into the effects of changes in various costs.

References

- [1] C. Alzaman, A. Ramudhin, and A. A. Bulgak. Gradient method in solving nonlinear cost of quality functions in supply chain network design. *International Journal of Management Science and Engineering Management*, 5(6):411–421, 2010.
- [2] I. P. Androulakis, C. D. Maranas, and C. A. Floudas. α bb: A global optimization method for general constrained nonconvex problems. *Journal of Global Optimization*, 7(4):337–363, 1995.
- [3] P. Belotti. *COUENNE: a user's manual*. Department of Mathematical Sciences, Clemson University.
- [4] J. Campanella. *Principles of Quality Costs: Principles, Implementation, and Use*. ASQ Quality Press, Milwaukee, WI, 1999.
- [5] K. K. Castillo-Villa, N. R. Smith, and J. L. Simonton. Heuristic procedure for a combinatorial optimization problem in supply chain design incorporating cost of quality. *Biomedical Soft Computing and Human Sciences*, 17(2):19–26, 2011.

- [6] P. B. Crosby. *Quality is Free*. McGraw-Hill, New York, NY, 1979.
- [7] A. V. Feigenbaum. Total quality control. *Harvard Business Review*, 34(6):93–101, 1956.
- [8] F. Glover and E. Woolsey. Converting the 0-1 polynomial programming problem to a 0-1 linear program. *Operations Research*, 22(1):180–182, 1974.
- [9] M. Hall and C. Tomkins. A cost of quality analysis of a building project: towards a complete methodology for design and build. *Construction Management and Economics*, 19(7):727–740, 2001.
- [10] C. D. Heagy. Determining optimal quality costs by considering costs of lost sales. *Journal of Cost Management for the Manufacturing Industry*, 5(Fall):64–72, 1991.
- [11] IBM. *IBM ILOG CPLEX Optimization Studio Getting Started with CPLEX*. Version 12 Release 4. 2011.
- [12] S. S. Kalagnanam and E. M. Matsumura. Cost of quality in an order entry department. *Journal of Cost Management*, 9(3):68–74, 1995.
- [13] Y. Lin and L. Schrage. The global solver in the lindo api. *Optimization Methods and Software*, 24(4–5):657–668, 2009.
- [14] D. M. Lundvall and J. M. Juran. Quality costs. J. M. Juran, F. M. Gryna, Jr., and R. S. Bingham, Jr. *Quality Control Handbook*, 5:1–22, McGraw-Hill, New York, NY, 3rd edition, 1974.
- [15] X. Ning and L.-G. Wang. Construction quality-cost trade-off using the Pareto-based ant colony optimization algorithm. *International Conference on Management and Service Science*, pages 1–4. IEEE, September 2009.
- [16] M. Oppermann, W. Sauer, and H. Wohlrabe. Optimization of quality costs. *Robotics and Computer-Integrated Manufacturing*, 19(1-2):135–140, 2003.
- [17] J. J. Plunkett and B. G. Dale. A review of the literature on quality-related costs. *International Journal of Quality & Reliability Management*, 4(1):40–52, 1987.
- [18] J. J. Plunkett and B. G. Dale. Quality-related costing: findings from an industry-based research study. *International Journal of Production Research*, 4(4):247–257, 1988.
- [19] M. Rahimi and H. Iranmanesh. On the discrete time, cost and quality trade-off problem. *Applied Mathematics and Computation*, 181(2):1305–1312, 2006.
- [20] M. Rahimi and H. Iranmanesh. Multi objective particle swarm optimization for a discrete time, cost and quality trade-off problem. *World Applied Sciences Journal*, 4(2):270–276, 2008.
- [21] A. Ramudhin, C. Alzaman, and A. A. Bulgak. Incorporating the cost of quality in supply chain design. *Journal of Quality in Maintenance Engineering*, 14(1):71–86, 2008.
- [22] R. E. Rosenthal. *GAMS – User’s Manual*, 2012.
- [23] N. V. Sahinidis. *BARON 11.4.0: Global Optimization of Mixed-Integer Nonlinear Programs*, User’s Manual, 2012.

- [24] A. Schiffauerova and V. Thomson. A review of research on cost of quality models and best practices. *International Journal of Quality & Reliability Management*, 23(6):647–669, 2006.
- [25] A. M. Schneiderman. Optimum quality costs and zero defects: Are they contradictory concepts? *Quality Progress*, pp. 28–31, November 1986.
- [26] J. W. Schoonard and S. J. Boies. Short-type: A behavioral analysis of typing and text entry. *Human Factors*, 17(2):203–214, 1975.
- [27] J. K. Shank and V. Govindarajan. Measuring the cost of quality: A strategic cost management perspective. *Journal of Cost Management*, 8(2):5–17, 1994.
- [28] L. U. Taticonda and R. J. Taticonda. Measuring and reporting the cost of quality. *Production & Inventory Management Journal*, 37:1–7, 1996.
- [29] M. Tawarmalani and N. V. Sahinidis. A polyhedral branch-and-cut approach to global optimization. *Mathematical Programming, Series B*, 103(2):225–249, 2005.
- [30] A. R. T. Williams, A. van der Wiele, and B. G. Dale. Quality costing: a management review. *International Journal of Management Reviews*, 1(4):441–460, 1999.
- [31] M. M. Yasin, A. J. Czuchry, J. J. Dorsch, and M. Small. In search of an optimal cost of quality: an integrated framework of operational efficiency and strategic effectiveness. *Journal of Engineering and Technology Management*, 16(2):171–189, 1999.