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ORIGINAL ARTICLE

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Semi-Markov modulated Poisson process: probabilistic and statistical analysis

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Abstract We consider a Poisson process that is modulated in such a way that the arrival rate at any time depends on the state of a semi-Markov process. This presents an interesting generalization of Poisson processes with important implications in real life applications. Our analysis concentrates on the transient as well as the long term behaviour of the arrival count and the arrival time processes. We discuss probabilistic as well as statistical issues related to various quantities of interest.

Keywords Poisson process · Semi-Markov modulation · Probabilistic analysis · Bayesian analysis

1 Introduction

Poisson process is perhaps the most widely used stochastic process in operations research and management sciences. It often reflects, quite precisely, the stochastic structure of arrivals of customers at a given location. It also offers computational tractability with the simplicity of the Poisson distribution and the exponential distribution of the interarrival times. Therefore, it has been one of the prime modelling tools in both the theory and practice of queues, among several other areas. In its simplest standard form, it is a right-continuous process with independent and stationary increments that increases by jumps of magnitude 1 only. This implies that

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the total number of arrivals in any time interval $[t, t+s)$ has the Poisson distribution with parameter λs where λ is the rate at which arrivals occur at any point in time. Despite widespread applicability, this is often unrealistic since the arrival rate is not necessarily constant in many cases. If the stationarity assumption on the increments is dropped, we have so-called nonstationary Poisson processes which can be transformed to ordinary Poisson processes by the well-known uniformization technique involving a deterministic time change operation. As such, the arrival rate at any time t is $\lambda(t)$ so that it is not a constant but a deterministic function of time. Still, one can think of many cases where this generalization fails to provide a sufficiently good approximation. This is especially true if the arrival rate at any time is also random.

One approach to attack this problem has been to represent the arrival rate function by a stationary stochastic process Z. This is known as the doubly stochastic Poisson process introduced by Cox (1955), and several variations of this idea are studied by others like Kingman (1964) and Berman (1981) to name a few. Research along this direction analyze the statistical properties of the modulated Poisson process through those of the modulating process Z.

In the present setting, we aim to describe this relationship further by supposing that the arrival rate process is modulated by a stochastic process Y with a discrete state space E such that

$$Z_t = \lambda(Y_t) \tag{1}$$

for all $t \geq 0$ where $\lambda(i)$ is the rate of arrivals when the state of Y is i. Here, we call Y the environmental process that affects and modulates the arrival rate of the Poisson process. This idea follows that presented in detail by Özekici (1996) who discusses the use of an environmental process as a source of random variations in the model parameters, as well as stochastic dependence among the model components of complex systems. Although the presentation in this paper focuses on a Poisson arrival process with a randomly varying arrival rate, the model described by (1) is more general with applications in a number of different areas. The reader is further referred to and Çınlar and Özekici (1987) for reliability models in random environments, and Özekici and Soyer (2003) for an application in software reliability engineering. Neuts (1978) and Prabhu and Zhu (1989) discuss modulated queueing models. Inventory models that are modulated by randomly changing economic environments are in a number of papers including Song and Zipkin (1993) and Erdem and Özekici (2002).

In section 2, we describe our semi-Markov modulated Poisson process model more precisely by introducing the relevant notation and terminology. Characterizations on the transient behaviour of the processes involved will be provided in section 3 and section 4 will be devoted to ergodic analysis. In section 5, we consider the Markov modulated Poisson process and present its Bayesian analysis in section 6. An illustration of our results is given in section 7.

2 Modulated Poisson model

Let N be a modulated Poisson process such that N_t depicts the total number of arrivals until time t. The modulation is done via an environmental process Y with a discrete state space E where Y_t represents the state of the environment at time t.

The rate of arrivals at time t is $\lambda(Y_t)$ for some arrival rate vector λ defined on E. We suppose that while the environment is at state i arrivals occur according to an ordinary Poisson process with rate $\lambda(i)$. To be more precise,

$$P[N_t = k|Y] = \frac{e^{-A_t} A_t^k}{k!}, \tag{2}$$

where

$$A_t = \int_0^t \lambda(Y_s) ds \tag{3}$$

for all $k = 0, 1, \ldots$ and $t \geq 0$.

It also follows from this construction that, given Y, N is a nonstationary Poisson process with expectation or mean value function $E[N_t|Y] = A_t$. Defining T to be the arrival time process so that T_n is the time of the nth arrival, we have the conditional interarrival time distribution

$$P[T_{n+1} - T_n > t|Y, T_n] = e^{-(A_{T_n+t} - A_{T_n})}. \tag{4}$$

The modulated process reduces to the ordinary Poisson process with rate λ if the arrival rate vector is $\lambda(i) = \lambda$ independent of the environment for all i. In this case, $A_t = \lambda t$ deterministically.

The arrival process N can be studied via the additive functional A of Y. In particular, (2) and (4) directly yield

$$P[N_t = k] = E\left[\frac{e^{-A_t} A_t^k}{k!}\right] \tag{5}$$

and

$$P[T_{n+1} - T_n > t|T_n] = E[e^{-(A_{T_n+t} - A_{T_n})}]. \tag{6}$$

Therefore, the probability law of A, thus that of the environmental process Y, will play an important role in our analysis of N and T. The discrete time version of this setting, for Bernoulli processes, is analyzed by Özekici (1997) and Özekici and Soyer (2003) where the success probability depends on the state of a Markov chain in each period.

We assume that Y is the minimal semi-Markov process associated with a Markov renewal process (X, S) with some semi-Markov kernel Q. In other words, X_n is the nth state visited by the environmental process and S_n is the time of this visit such that

$$Y_t = X_n \quad \text{whenever } S_n \leq t < S_{n+1}. \tag{7}$$

Moreover, the semi-Markov kernel gives

$$Q(i, j, t) = P[X_{n+1} = j, S_{n+1} - S_n \leq t|X_n = i] \tag{8}$$

for all $i, j \in E$ and $t \geq 0$. The process X is a Markov chain with transition matrix $Q(i, j, +\infty)$ and the process Y spends a random amount of time with distribution function

$$F_i(t) = P[S_1 \leq t|Y_0 = i] = \sum_{j \in E} Q(i, j, t) \tag{9}$$

in state i. Note that S_1 is the time of first jump of Y and it represents the amount of time spent in state i if $Y_0 = i$. This random variable and its distribution will play a crucial role in the following discussions. The Markov renewal kernel defined as

$$R(i, j, t) = \sum_{n=0}^{+\infty} Q^n(i, j, t) \tag{10}$$

gives the expected number of visits to the environmental state j until time t if the initial state is i where Q^n is the n-fold convolution of Q by itself. We will see that it plays a critical role in the analysis of Y and A. Finally, we will let P_t denote the transition function of Y so that

$$P_t(i, j) = P[Y_t = j | Y_0 = i]. \tag{11}$$

It is known that the transition function can be characterized by a Markov renewal equation the solution of which is

$$P_t(i, j) = \int_{[0,t)} R(i, j, ds)\overline{F}_j(t - s), \tag{12}$$

where $\overline{F}_j = 1 - F_j$ is the survival function corresponding to F_j. For notation, terminology and results on Markov renewal theory, we refer our reader to Çınlar (1975).

3 Transient characterizations

Note that although N is conditionally a nonstationary Poisson process given Y, it no longer enjoys the independence of its increments without conditioning. Therefore, the major requirement for Poisson processes is not satisfied anymore. However, there are still many interesting characterizations pertaining to N and T. In particular, the generating function of N_t can be written as

$$\begin{aligned} E[\alpha^{N_t}] &= \sum_{k=0}^{+\infty} \alpha^k E\left[\frac{e^{-A_t} A_t^k}{k!}\right] = E\left[e^{-A_t} \sum_{k=0}^{+\infty} \frac{(\alpha A_t)^k}{k!}\right] \\ &= E[e^{-(1-\alpha)A_t}] = E[e^{-A_t^\alpha}] \end{aligned} \tag{13}$$

for $0 \leq \alpha \leq 1$ where

$$A_t^\alpha = (1 - \alpha)A_t = (1 - \alpha)\int_0^t \lambda(Y_s)ds = \int_0^t \lambda_\alpha(Y_s)\mathrm{d}s, \tag{14}$$

and λ_α is an arrival rate vector defined as $\lambda_\alpha(i) = (1 - \alpha)\lambda(i)$ for each $i \in E$. Therefore, $\lambda_\alpha \leq \lambda$ represents a scaled arrival rate vector with scaling factor $(1-\alpha)$.

Let N^α be a modulated Poisson process with arrival rate vector λ_α and T^α be the corresponding arrival time process. It now follows directly from (4) that

$$E[\alpha^{N_t}] = E\left[e^{-A_t^\alpha}\right] = P\left[T_1^\alpha > t\right] \tag{15}$$

for all $t \geq 0$ and $0 \leq \alpha \leq 1$. In other words, if the distribution of the time to the first arrival of a modulated Poisson process is known, then one can obtain a complete characterization on the distribution of N_t for all t. For a given λ, if we let $\overline{G}_\lambda(t) = P[T_1 > t]$, then (15) can equivalently be written as

$$E\left[\alpha^{N_t}\right] = P\left[T_1^\alpha > t\right] = \overline{G}_{\lambda_\alpha}(t). \tag{16}$$

If N is an ordinary Poisson process with a constant arrival rate λ, then N_t has the Poisson distribution with mean λt so that

$$E\left[\alpha^{N_t}\right] = e^{-(1-\alpha)\lambda t}. \tag{17}$$

This is in the form characterized for a modulated Poisson process by (16) where $\overline{G}_\lambda(t) = e^{-\lambda t}$ is the exponential survival function corresponding to the time of the first arrival. This exponential distribution of the time to the first arrival provides a complete characterization of the distribution of N_t via (17) so that its generating function is $\overline{G}_{\lambda_\alpha}(t) = e^{-(1-\alpha)\lambda t}$. It is important to note that the same is true for a modulated Poisson process via (16), but now the distribution of the time to the first arrival is not necessarily exponential. Whatever it is, the scaled survival function $\overline{G}_{\lambda_\alpha}(t)$ gives the generating function of N_t.

Since the modulating process involves a Markov renewal process, characterizations on the transient behaviour of several quantities of interest can be obtained using Markov renewal theory. To simplify our notation, we will set $E_i[Z] \equiv E[Z|Y_0 = i]$ and $P_i[B] = P[B|Y_0 = i]$ for any random variable Z and any event B. We will analyze the conditional survival function $P_i[T_1 > t]$ of the time to the first arrival for a given arrival rate vector λ. This function can be used in a trivial manner to compute $\overline{G}_\lambda(t) = P[T_1 > t] = \sum_{i \in E} P[Y_0 = i] P_i[T_1 > t]$ using the initial distribution of Y.

Renewal theoretic discussions yield

$$\begin{aligned} P_i[T_1 > t] &= P_i[T_1 > t, S_1 > t] + P_i[T_1 > t, S_1 \leq t] \\ &= e^{-\lambda(i)t}\overline{F}_i(t) + \sum_{j \in E} \int_{[0,t)} Q(i,j,ds) e^{-\lambda(i)s} P_j[T_1 > t-s] \\ &= e^{-\lambda(i)t}\overline{F}_i(t) + \sum_{j \in E} \int_{[0,t)} Q_\lambda(i,j,ds) P_j[T_1 > t-s], \end{aligned} \tag{18}$$

where $Q_\lambda(i,j,ds) = Q(i,j,ds)e^{-\lambda(i)s}$ is also a semi-Markov kernel. Therefore, (18) is a Markov renewal equation of the form $f = g_\lambda + Q_\lambda * f$ where $g_\lambda(i,t) = e^{-\lambda(i)t}\overline{F}_i(t)$. It has the unique solution $f = R_\lambda * g_\lambda$, or

$$P_i[T_1 > t] = R_\lambda * g_\lambda(i,t) = \sum_{j \in E} \int_{[0,t)} R_\lambda(i,j,ds) e^{-\lambda(j)(t-s)} \overline{F}_j(t-s), \tag{19}$$

where $R_\lambda = \sum_{n=0}^{+\infty} Q_\lambda^n$ is the Markov renewal kernel corresponding to Q_λ.

This conditional survival function also gives the conditional generating function of N_t as

$$E_i\left[\alpha^{N_t}\right] = P_i\left[T_1^\alpha > t\right] = R_{\lambda_\alpha} * g_{\lambda_\alpha}(i,t)$$
$$= \sum_{j\in E}\int_{[0,t)} R_{\lambda_\alpha}(i,j,ds)e^{-(1-\alpha)\lambda(j)(t-s)}\overline{F}_j(t-s) \qquad (20)$$

after repeating the arguments for (13), (14), (15) and (16). Another quantity of interest is

$$E_i[N_t] = E_i[N_t; S_1 > t] + E_i[N_t; S_1 \le t]$$
$$= \lambda(i)t\overline{F}_i(t) + \sum_{j\in E}\int_{[0,t)} Q(i,j,ds)(\lambda(i)s + E_j[N_{t-s}])$$
$$= \lambda(i)t\overline{F}_i(t) + \lambda(i)\int_{[0,t)} F_i(ds)s + \sum_{j\in E}\int_{[0,t)} Q(i,j,ds)E_j[N_{t-s}]$$
$$= \lambda(i)\int_0^t \overline{F}_i(s)ds + \sum_{j\in E}\int_{[0,t)} Q(i,j,ds)E_j[N_{t-s}], \qquad (21)$$

where the last equality follows by noting that $t\overline{F}_i(t) + \int_{[0,t)} F_i(ds)s = E_i[\min\{S_1,t\}] = \int_0^t \overline{F}_i(s)ds$.

Note that (21) is in fact a Markov renewal equation of the form $f = h_\lambda + Q * f$ where $h_\lambda(i,t) = \lambda(i)\int_0^t \overline{F}_i(s)ds$ and it has the unique solution $f = R * h_\lambda$. In open form, this solution is written as

$$E_i[N_t] = R * h_\lambda(i,t) = \sum_{j\in E}\int_{[0,t)} R(i,j,ds)\lambda(j)\int_0^{t-s}\overline{F}_j(s)ds. \qquad (22)$$

Let M_t denote the number of arrivals until time t since the last environment change before time t, i.e., $M_t = N_t - N_{S_n}$ whenever $S_n \le t < S_{n+1}$. It is clear that M is a semi-regenerative process with $M_{S_n} = 0$ for all n. Moreover, its conditional generating function $E_i[\alpha^{M_t}]$ satisfies the Markov renewal equation

$$E_i\left[\alpha^{M_t}\right] = E_i\left[\alpha^{M_t}; S_1 > t\right] + E_i\left[\alpha^{M_t}; S_1 \le t\right]$$
$$= e^{-(1-\alpha)\lambda(i)t}\overline{F}_i(t) + \sum_{j\in E}\int_{[0,t)} Q(i,j,ds)E_j\left[\alpha^{M_{t-s}}\right], \qquad (23)$$

which has the unique solution

$$E_i\left[\alpha^{M_t}\right] = \sum_{j\in E}\int_{[0,t)} R(i,j,ds)e^{-(1-\alpha)\lambda(j)(t-s)}\overline{F}_j(t-s). \qquad (24)$$

A similar discussion gives the characterization

$$E_i[M_t] = \sum_{j\in E}\int_{[0,t)} R(i,j,ds)\lambda(j)(t-s)\overline{F}_j(t-s). \qquad (25)$$

The analysis provided in this section for various quantities of interest demonstrate how Markov renewal theory can be used to characterize the transient behaviour of the semi-Markov modulated Poisson model. The difficulty with most of the results presented here is in determining R or R_λ. The formulas are not computationally tractable but they can be used to obtain approximations. The long term behaviour of the model, however, can be more easily analyzed via computationally tractable formulas.

4 Ergodic analysis

Suppose that both X and Y are ergodic processes with limiting distributions $\nu(j) = \lim_{n\to+\infty} P[X_n = j]$ and $\pi(j) = \lim_{t\to+\infty} P[Y_t = j]$. This implies that ν is the unique solution of $\nu = \nu P$ with the normalizing condition $\sum_{i\in E} \nu(i) = 1$ where $P(i, j) = Q(i, j, +\infty)$ is the transition matrix of the embedded Markov chain X. It is well-known that

$$\lim_{t\to+\infty} R * g(i, t) = \frac{\sum_{j\in E} \nu(j) \int_0^{+\infty} g(j, u)du}{\sum_{i\in E} \nu(i)m(i)}, \tag{26}$$

where $m(j) = E_j[S_1]$ provided that g is directly Riemann integrable. This will indeed be true in all of the cases discussed below. For example, the application of (26) to the transition function in (12) gives the limiting distribution

$$\pi(j) = \lim_{t\to+\infty} P_t(i, j) = \frac{\nu(j)m(j)}{\sum_{k\in E} \nu(k)m(k)} \tag{27}$$

as the normalization of $\{\nu(j)m(j)\}$.

It is quite clear that $\lim_{t\to+\infty} E_i[\alpha^{N_t}] = 0$ for $\alpha < 1$ and $\lim_{t\to+\infty} E_i [N_t] = +\infty$. But, the application of (26) to (24) and (25) respectively yields the explicit formula.

$$\begin{aligned}
\lim_{t\to+\infty} E_i\left[\alpha^{M_t}\right] &= \lim_{t\to+\infty} \sum_{j\in E} \int_{[0,t)} R(i, j, ds)e^{-(1-\alpha)\lambda(j)(t-s)}\overline{F}_j(t-s) \\
&= \frac{\sum_{j\in E} \nu(j) \int_0^{+\infty} e^{-(1-\alpha)\lambda(j)u}\overline{F}_j(u)du}{\sum_{k\in E} \nu(k)m(k)} \\
&= \frac{\sum_{j\in E} \nu(j)[(1-\mathcal{F}_j((1-\alpha)\lambda(j)))/(1-\alpha)\lambda(j)]}{\sum_{k\in E} \nu(k)m(k)} \qquad (28) \\
&= \sum_{j\in E} \pi(j)\left[(1-\mathcal{F}_j((1-\alpha)\lambda(j)))/(1-\alpha)\lambda(j)m(j)\right] \\
&= \sum_{j\in E} \pi(j)\mathcal{M}_j((1-\alpha)\lambda(j)) \qquad (29)
\end{aligned}$$

where $\mathcal{M}_j(\alpha) = (1-\mathcal{F}_j(\alpha))/\alpha m(j)$. Here, $\mathcal{F}_j$ is the Laplace transform of F_j. One can also show that $\mathcal{M}_j(\alpha) = E[e^{-\alpha\hat{S}_j}]$ is the Laplace transform of the generic random variable $\hat{S}_j$ that represents the stationary backward recurrence time, or age, of a renewal process with interrenewal distribution F_j. It also follows that $\mathcal{M}_j((1-\alpha)\lambda(j))$ is the generating function of the number of arrivals until time $\hat{S}_j$ of a Poisson process with arrival rate $\lambda(j)$. In light of this interpretation, the meaning of (29) is self-explanatory. In this derivation, (28) follows by noting that

$$\begin{aligned}\int_0^{+\infty} e^{-(1-\alpha)\lambda(j)u}\overline{F}_j(u)du &= \int_0^{+\infty} e^{-(1-\alpha)\lambda(j)u}du \int_{(u,+\infty)} F_j(ds)\\ &= \int_0^{+\infty} F_j(ds)\int_0^s e^{-(1-\alpha)\lambda(j)u}du\\ &= \int_0^{+\infty} \frac{F_j(ds)(1-e^{-(1-\alpha)\lambda(j)s})}{(1-\alpha)\lambda(j)}\\ &= \frac{1-\mathcal{F}_j((1-\alpha)\lambda(j))}{(1-\alpha)\lambda(j)}.\end{aligned}$$

By similar reasoning,

$$\begin{aligned}\lim_{t\to+\infty} E_i[M_t] &= \lim_{t\to+\infty}\sum_{j\in E}\int_{[0,t)} R(i,j,ds)\lambda(j)(t-s)\overline{F}_j(t-s)\\ &= \frac{\sum_{j\in E}\nu(j)\int_0^{+\infty}\lambda(j)u\overline{F}_j(u)du}{\sum_{k\in E}\nu(k)m(k)}\\ &= \frac{\sum_{j\in E}\nu(j)\lambda(j)E_j\left[S_1^2\right]}{2\sum_{k\in E}\nu(k)m(k)} \qquad (30)\\ &= \sum_{j\in E}\pi(j)\lambda(j)\frac{E_j\left[S_1^2\right]}{2m(j)} = \sum_{j\in E}\pi(j)\lambda(j)\hat{m}(j), \qquad (31)\end{aligned}$$

where $\hat{m}(j) = E[\hat{S}_j]$. Note that (30) follows from

$$\begin{aligned}\int_0^{+\infty}\lambda(j)u\overline{F}_j(u)du &= \int_0^{+\infty}\lambda(j)udu\int_{(u,+\infty)}F_j(ds)\\ &= \lambda(j)\int_0^{+\infty}F_j(ds)\int_0^s udu\\ &= \lambda(j)\int_0^{+\infty}F_j(ds)\frac{s^2}{2} = \frac{\lambda(j)E_j\left[S_1^2\right]}{2}.\end{aligned}$$

Although N_t increases without bound like an ordinary Poisson process, we can characterize its behaviour for large t using Markov renewal theory.

Theorem 1 *Let* $\hat{\lambda} = \sum_{j\in E}\pi(j)\lambda(j)$, *then*

$$E_i[N_t - \hat{\lambda}t] = \sum_{j\in E}\int_{[0,t)} R(i,j,ds)(\lambda(j)-\hat{\lambda})\int_0^{t-s}\overline{F}_j(u)du, \qquad (32)$$

so that

$$\lim_{t\to+\infty} E_i[N_t - \hat{\lambda}t] = \sum_{j\in E} \pi(j)(\hat{\lambda} - \lambda(j))\hat{m}(j), \tag{33}$$

and

$$\lim_{t\to+\infty} \frac{E_i[N_t]}{t} = \hat{\lambda} \tag{34}$$

for all i.

Proof Let $n(i,t) = E_i[N_t - \hat{\lambda}t]$, then conditioning on S_1 gives

$$\begin{aligned}
n(i,t) &= E_i\left[N_t - \hat{\lambda}t;\, S_1 > t\right] + E_i[N_t - \hat{\lambda}t;\, S_1 \le t] \\
&= (\lambda(i) - \hat{\lambda})t\overline{F}_i(t) \\
&\quad + \sum_{j\in E} \int_{[0,t)} Q(i,j,ds)(\lambda(i)s + E_j[N_{t-s}] - \hat{\lambda}s - \hat{\lambda}(t-s)) \\
&= (\lambda(i) - \hat{\lambda})t\overline{F}_i(t) + \sum_{j\in E} \int_{[0,t)} Q(i,j,ds)(\lambda(i) - \hat{\lambda})s \\
&\quad + \sum_{j\in E} \int_{[0,t)} Q(i,j,ds)n(j,t-s) \\
&= (\lambda(i) - \hat{\lambda}) \int_0^t \overline{F}_i(s)ds + \sum_{j\in E} \int_{[0,t)} Q(i,j,ds)n(j,t-s).
\end{aligned} \tag{35}$$

This is a Markov renewal equation of the form $n = g + R * n$ where $g(i,n) = (\lambda(i) - \hat{\lambda}) \int_0^t \overline{F}_i(s)ds$ and the unique solution is given by (32). Using (26), the limiting value is

$$\begin{aligned}
\lim_{t\to+\infty} E_i[N_t - \hat{\lambda}t] &= \frac{\lim_{u\to+\infty} \sum_{j\in E} \nu(j)(\lambda(j) - \hat{\lambda}) \int_0^u dt \int_0^t \overline{F}_j(s)ds}{\sum_{k\in E} \nu(k)m(k)} \\
&= \lim_{u\to+\infty} \sum_{j\in E} \pi(j)(\lambda(j) - \hat{\lambda}) \int_0^u dt \int_0^t \left[\frac{\overline{F}_j(s)}{m(j)}\right] ds \\
&= \lim_{u\to+\infty} \sum_{j\in E} \pi(j)(\lambda(j) - \hat{\lambda}) \int_0^u dt H_j(t)
\end{aligned}$$

where H_j is the stationary backward recurrence time distribution corresponding to F_j. It is well-known that the mean of this distribution is $\hat{m}(j) = \int_0^{+\infty} \overline{H}_j(t)dt = E_j[S_1^2]/2E_j[S_1]$. Therefore,

$$\lim_{t\to+\infty} E_i[N_t - \hat{\lambda}t] = \lim_{u\to+\infty} \sum_{j\in E} \pi(j)(\lambda(j) - \hat{\lambda}) \int_0^u dt[1 - \overline{H}_j(t)]$$
$$= \lim_{u\to+\infty} \sum_{j\in E} \pi(j)(\lambda(j) - \hat{\lambda}) \left[u - \int_0^u \overline{H}_j(t)dt\right]$$
$$= \sum_{j\in E} \pi(j)(\hat{\lambda} - \lambda(j)) \frac{E_j\left[S_1^2\right]}{2E_j[S_1]},$$

which finally leads to the desired result in (33). The critical point in this derivation is that $\sum_{j\in E} \pi(j)(\lambda(j) - \hat{\lambda}) = 0$ by the way $\hat{\lambda}$ is defined. Since the limit in (33) is equal to a constant, we also have $\lim_{t\to+\infty} E_i[N_t]/t = \hat{\lambda}$ trivially. □

It is interesting to note that $E_i[N_t - \hat{\lambda}t] = 0$ for all t if N is an ordinary Poisson process with constant arrival rate $\lambda(i) = \hat{\lambda}$ in all environments. Although this is not true for the modulated process, $E_i[N_t]$ behaves like $\hat{\lambda}t$ for large t in the manner prescribed by (32). We can call $\hat{\lambda}$ the average arrival rate where the average is taken with respect to the limiting distribution π of Y. The justification for this is provided by (34).

5 Markov modulation

If the environmental process Y is a Markov process, then most of the computational results obtained in the previous sections can be simplified. Note that the transient results involve Markov renewal characterizations that require evaluation of convolutions. Although there are procedures to approximate them, this clearly poses some computational difficulties. Throughout the remainder of this paper we suppose that Y is a Markov process to obtain more computationally tractable results. This implies that

$$Q(i, j, t) = P(i, j)(1 - e^{-\mu(i)t}), \tag{36}$$

where P is the transition matrix of the embedded Markov chain X and μ is the vector of jump rates. More precisely, if the process Y is in some state i, then it stays there for an exponentially distributed amount of time with rate $\mu(i)$ and then jumps to some other state j with probability $P(i, j)$. It also follows that

$$F_i(t) = 1 - e^{-\mu(i)t}, \tag{37}$$

and the generator of the Markov process Y is given by the matrix

$$G(i, j) = \begin{cases} -\mu(i), & \text{if } j = i, \\ \mu(i)P(i, j), & \text{if } j \neq i. \end{cases} \tag{38}$$

We now define another process Y^λ such that

$$Y_t^\lambda = \begin{cases} Y_t, & \text{if } t < T_1, \\ \Delta, & \text{if } t \geq T_1, \end{cases} \tag{39}$$

where T_1 is the time of the first arrival. While the environment is in state i, the time of an arrival has the exponential distribution with rate $\lambda(i)$. It is clear that Y^{λ} is also a Markov process on the extended state space $E_{\Delta} = E \cup \{\Delta\}$ and it is obtained by "stopping" the Markov process Y as soon as an arrival occurs. Here, Δ is an absorbing state where the process is dumped to as soon as it is stopped. The transition matrix of the embedded Markov chain is now extended as

$$P_{\lambda}(i, j) = \begin{cases} \frac{\mu(i)}{\mu(i)+\lambda(i)} P(i, j), & \text{if } i, j \in E, \\ \frac{\lambda(i)}{\mu(i)+\lambda(i)}, & \text{if } i \in E, j = \Delta, \\ 1, & \text{if } i, j = \Delta, \end{cases} \tag{40}$$

and the transition rate vector is

$$\mu_{\lambda}(i) = \begin{cases} \mu(i) + \lambda(i), & \text{if } i \in E, \\ 0, & \text{if } i = \Delta. \end{cases} \tag{41}$$

If we let the matrix $G_{\lambda}(i, j) = \mu_{\lambda}(i)(P_{\lambda}(i, j) - I(i, j))$ denote the generator of Y^{λ}, then it is well-known that the transition function $P_t^{\lambda}(i, j) = P[Y_t^{\lambda} = j | Y_0^{\lambda} = i]$ for all $i, j \in E_{\Delta}$ is given by the matrix-exponential solution

$$P_t^{\lambda} = e^{G_{\lambda} t} = \sum_{n=0}^{+\infty} \frac{t^n}{n!} G_{\lambda}^n. \tag{42}$$

A further simplification is obtained by noting that

$$G_{\lambda}(i, j) = G(i, j) - \Lambda(i, j) \tag{43}$$

for all $i, j \in E$ where Λ is a diagonal matrix defined as

$$\Lambda(i, j) = \begin{cases} \lambda(i), & \text{if } j = i, \\ 0, & \text{if } j \neq i. \end{cases} \tag{44}$$

Since $G_{\lambda}(\Delta, j) = 0$ and $G_{\lambda}(i, \Delta) = \lambda(i)$ for all $i \in E$ and $j \in E_{\Delta}$, we can rewrite (42) as

$$P_t^{\lambda}(i, j) = e^{G_{\lambda} t}(i, j) = e^{(G-\Lambda)t}(i, j) \tag{45}$$

for all $i, j \in E$.

Now, note that our construction of Y^{λ} implies

$$T_1 = \inf \left\{ t \geq 0; Y_t^{\lambda} = \Delta \right\}, \tag{46}$$

and T_1 is the first-passage-time to the absorbing state Δ. So, it has a phase-type distribution and, in particular,

$$P_i[T_1 > t] = P_i \left[Y_t^{\lambda} \in E \right] = \sum_{j \in E} P_t^{\lambda}(i, j) = \sum_{j \in E} e^{(G-\Lambda)t}(i, j), \tag{47}$$

which is the simplified version of (19). Note that in reliability applications where a device fails exponentially with a failure rate that depends on the randomly changing environment, (47) gives the survival function. In this case, the mean time to failure (MTTF) is also another quantity of interest. Using the Markov property, it can be computed by solving the system of linear equations

$$E_i[T_1] = \mu_\lambda(i)^{-1} + \sum_{j \in E} P_\lambda(i, j) E_j[T_1] \tag{48}$$

for $i \in E$ so that the explicit solution is

$$E_i[T_1] = \sum_{j \in E} [I - P_\lambda]^{-1}(i, j)\mu_\lambda(j)^{-1}. \tag{49}$$

Now, (20) reduces to

$$\begin{aligned} E_i\left[\alpha^{N_t}\right] &= P_i\left[T_1^\alpha > t\right] = P_i\left[Y_t^{\lambda_\alpha} \in E\right] \\ &= \sum_{j \in E} P_t^{\lambda_\alpha}(i, j) = \sum_{j \in E} e^{(G-(1-\alpha)\Lambda)t}(i, j) \end{aligned} \tag{50}$$

to yield an explicit characterization for the conditional generating function of N_t since $\lambda_\alpha(i) = (1-\alpha)\lambda(i)$.

In ergodic analysis, we now have $m(j) = E_j[S_1] = 1/\mu(j)$ and

$$\pi(j) = \frac{\nu(j)/\mu(j)}{\sum_{k \in E}(\nu(k)/\mu(k))} \tag{51}$$

replaces (27). Since $\mathcal{F}_j(\alpha) = \mu(j)/(\alpha + \mu(j))$ is the Laplace transform of the exponential distribution, (29) simplifies to

$$\lim_{t \to +\infty} E_i\left[\alpha^{M_t}\right] = \sum_{j \in E} \pi(j)\left[\frac{\mu(j)}{(1-\alpha)\lambda(j) + \mu(j)}\right], \tag{52}$$

while (31) becomes

$$\lim_{t \to +\infty} E_i[M_t] = \sum_{j \in E} \pi(j)\left[\frac{\lambda(j)}{\mu(j)}\right] \tag{53}$$

by taking $\hat{m}(j) = m(j) = 1/\mu(j)$ since the stationary backward recurrence time distribution of the exponential distribution is itself.

Finally, Theorem 1 implies

$$\lim_{t \to +\infty} E_i[N_t - \hat{\lambda}t] = \sum_{j \in E} \pi(j)\left[\frac{\hat{\lambda} - \lambda(j)}{\mu(j)}\right], \tag{54}$$

and

$$\lim_{t \to +\infty} \frac{E_i[N_t]}{t} = \hat{\lambda} = \sum_{j \in E} \pi(j)\lambda(j). \tag{55}$$

Differentiating the moment generating function (50) with respect to α and evaluating it at α =1 as in Fischer and Meier-Hellstern (1992), we can also obtain the expected number of arrivals until time t as

$$E_i[N_t] = \hat{\lambda}t + \sum_{j \in E}\left(\left[e^{Gt} - I\right][G + \Pi]^{-1}\right)(i, j)\lambda(j), \tag{56}$$

where $\Pi(i, j) = \pi(j)$.

6 Bayesian analysis of the Markov modulated process

The model and the analysis presented above are based on the assumption that the parameters are specified. In this section, we will present a Bayesian analysis of the Markov modulated model by describing the uncertainty about the parameters probabilistically via some prior distributions. In particular, we will assume independent gamma priors on each $\lambda(i)$ with shape parameter $a(i)$ and scale parameter $b(i)$, denoted as $\lambda(i) \sim$ Gamma $(a(i),\ b(i))$ for all $i \in E$. Similarly, independent gamma priors are assumed for the rates $\mu(i)$ as $\mu(i) \sim$ Gamma $(c(i),\ d(i))$ for all $i \in E$. The prior distribution for the transition matrix of the embedded Markov chain X is specified by assuming that the ith row $P_i = \{P(i, j);\ j \in E\}$ has a Dirichlet prior

$$p(P_i) \propto \prod_{j \in E} P(i, j)^{\alpha^i_j - 1} \tag{57}$$

denoted as Dirichlet $\{\alpha^i_j;\ j \in E\}$ and P_i's are independent for all $i \in E$. Furthermore, it is assumed that apriori $\lambda(i)$'s, $\mu(i)$'s, and P_i's are independent of each other. We define $\Theta = \{(\lambda(i), \mu(i), P_i);\ i \in E\}$ and denote the joint prior distribution of Θ by $p(\Theta)$.

6.1 Prior analysis

We note that the generator matrix G of the environmental process Y and the rate matrix Λ of the Poisson process are functions of Θ . Thus, the results presented under the Markov modulation are conditional on the unknown vector Θ. Prior to observing any data we can make predictions for quantities such as the time of the first arrival and for the expected number of arrivals until time t using the prior distribution $p(\Theta)$.

In obtaining the distribution of the time of the first arrival we consider (47) and rewrite it as

$$P[T_1 > t | Y_0 = i, \Theta] = P_i[T_1 > t | \Theta] = \sum_{j \in E} e^{(G(\Theta) - \Lambda(\Theta))t}(i, j). \tag{58}$$

Conditional on Θ, $e^{(G(\Theta) - \Lambda(\Theta))t}$ can be computed from the matrix exponential form using one of the available methods, for example, in Moler and van Loan (1978). Alternatively, if the eigenvalues of the matrix $H(\Theta) = G(\Theta) - \Lambda(\Theta)$ are distinct, we can obtain

$$e^{(G(\Theta) - \Lambda(\Theta))t} = A(\Theta) e^{D(\Theta)t} A(\Theta)^{-1}, \tag{59}$$

where $D(\Theta)$ is a diagonal matrix of the eigenvalues of $H(\Theta)$, and $A(\Theta)$ is the matrix of the eigenvectors corresponding to the distinct eigenvalues such that $H(\Theta) = A(\Theta) D(\Theta) A(\Theta)^{-1}$. Given the matrix exponential solution, prior to any data we can make probability statements as

$$P[T_1 > t | Y_0 = i] = \int P[T_1 > t | Y_0 = i, \Theta] p(\Theta) d\Theta, \tag{60}$$

which can be approximated via simulation as a Monte Carlo integral

$$P[T_1 > t|Y_0 = i] \approx \frac{1}{S}\sum_{s} P\left[T_1 > t|Y_0 = i, \Theta^{(s)}\right] \tag{61}$$

by generating S realizations from the prior distribution $p(\Theta)$.

Similarly to obtain the expected number arrivals until time t we rewrite (56) as

$$\begin{aligned} E[N_t|Y_0 = i, \Theta] &= E_i[N_t|\Theta] \\ &= \hat{\lambda}(\Theta)t + \sum_{j\in E}\left(\left[e^{G(\Theta)t} - I\right][G(\Theta) + \Pi(\Theta)]^{-1}\right)(i, j)\lambda(j) \end{aligned} \tag{62}$$

to note that the longrun rate $\hat{\lambda}$ and the limiting distribution matrix Π of the Y process are functions of the unknown parameters. For a given value of the Θ, (62) can be computed by evaluating (51) and (55). Then the prior expected number of arrivals is given by

$$E[N_t|Y_0 = i] = \int E[N_t|Y_0 = i, \Theta]p(\Theta)d\Theta, \tag{63}$$

which can be approximated as

$$E[N_t|Y_0 = i] \approx \frac{1}{S}\sum_{s} E\left[N_t|Y_0 = i, \Theta^{(s)}\right]. \tag{64}$$

Note that prior distributions for limiting probabilities (51) and the longrun rate (55) can also be obtained using similar Monte Carlo approximations.

6.2 Posterior analysis

Assume that the modulated Poisson process is observed for τ units of time during which K changes in the environmental process are observed. Furthermore, arrival times of the process under each environment as well as the environments and their durations are observed, that is, the observed data is given by

$$\mathcal{D} = \left\{(X_k, S_k), \left(T_1^k, T_2^k, \ldots, T_{M_k}^k\right);\quad k = 1, \ldots, K\right\}, \tag{65}$$

where T_j^k is the time of jth arrival during the kth environment and M_k is the number of arrivals during the kth environment. Moreover, X_k is the kth state visited by the environmental process while S_k is the time at which the environmental process enters the kth state.

We assume that the initial environment is $X_1 = i$ for some environment i and it starts at $S_1 = 0$. Since the results presented in previous sections are all conditional on some initial state i, the posterior analysis will be based on this arbitrary initialization. The likelihood function of $\Theta = \{(\lambda(i), \mu(i), P_i);\ i \in E\}$ given data $\mathcal{D}$ is

$$\mathcal{L}(\Theta|\mathcal{D}) = \mathcal{L}_K \left\{ \prod_{k=1}^{K-1} P(X_k, X_{k+1})\mu(X_k)e^{-\mu(X_k)(S_{k+1}-S_k)} \cdot \lambda(X_k)^{M_k}\, e^{-\lambda(X_k)(S_{k+1}-S_k)} \right\} \tag{66}$$

where $\mathcal{L}_K$ is the contribution of the Kth environment to the likelihood given by

$$\mathcal{L}_K = e^{-\mu(X_K)(\tau-S_K)}\lambda(X_K)^{M_K}\, e^{-\lambda(X_K)(\tau-S_K)}. \tag{67}$$

Given the independent Dirichlet priors, the posterior distribution of P_i's can be obtained as independent Dirichlets given by

$$(P_i|\mathcal{D}) \sim \text{Dirichlet}\left\{ \alpha_j^i + \sum_{k=1}^{K-1} 1(X_k = i, X_{k+1} = j);\ j \in E \right\}, \tag{68}$$

where $1(.)$ is the indicator function. Similarly, the posterior distributions of $\mu(i)$'s are obtained as independent gamma densities given by

$$(\mu(i)|\,\mathcal{D}) \sim \text{Gamma}\left(c(i) + \sum_{k=1}^{K-1} 1(X_k = i),\ d(i) + \sum_{k=1}^{K} (S_{k+1} - S_k)1(X_k = i) \right), \tag{69}$$

where $S_{K+1} = \tau$. We note that posteriori $\mu(i)$'s and P_i's are independent of $\lambda(i)$'s as well as each other. The posterior distributions of $\lambda(i)$'s are obtained as

$$(\lambda(i)|\,\mathcal{D}) \sim \text{Gamma}\left(a(i) + \sum_{k=1}^{K-1} M_k 1(X_k = i),\ b(i) + \sum_{k=1}^{K} (S_{k+1} - S_k)1(X_k = i) \right), \tag{70}$$

where $S_{K+1} = \tau$. Again $\lambda(i)$'s are independent of each other.

Once our uncertainty about Θ is revised to $p(\Theta|\mathcal{D})$, then we are interested in making posterior predictions for time to the next arrival via $P[\hat{T}_1 > t|\mathcal{D}]$, and for the expected number of arrivals via $E[\hat{N}_t|\mathcal{D}]$. Here, $\hat{T}_1$ is the time of the first arrival after time τ and $\hat{N}_t = N_{\tau+t} - N_\tau$ is the number of arrivals during $[\tau, \tau + t]$. Note that conditional on Θ, using the Markov property of the Y process and properties of the Poisson process, we can write

$$P[\hat{T}_1 > t|Y_0 = i, \mathcal{D}, \Theta] = P[T_1 > t|Y_\tau, \Theta], \tag{71}$$

and

$$E[\hat{N}_t|Y_0 = i, \mathcal{D}, \Theta] = E[N_{\tau+t} - N_\tau|Y_0 = i, \mathcal{D}, \Theta] = E[N_t|Y_\tau, \Theta], \quad (72)$$

where $Y_\tau = X_K$. Thus, using our previous results we have

$$P[T_1 > t|X_K, \Theta] = \sum_{j \in E} e^{(G(\Theta)-\Lambda(\Theta))t}(X_K, j) \quad (73)$$

where the matrix exponential form can be computed using previously discussed methods. Similarly, we have

$$E[N_t|X_K, \Theta] = \hat{\lambda}(\Theta)t + \sum_{j \in E} \left(\left[e^{G(\Theta)t} - I \right] [G(\Theta) + \Pi(\Theta)]^{-1} \right)(X_K, j)\lambda(j). \quad (74)$$

We can obtain the posterior prediction for time to the next arrival as

$$P[T_1 > t|\mathcal{D}] = \int P[T_1 > t|X_K, \Theta]p(\Theta|\mathcal{D})d\Theta \quad (75)$$

which can be approximated, using (73), as a Monte Carlo integral

$$P[T_1 > t|\mathcal{D}] \approx \frac{1}{S} \sum_s P\left[T_1 > t|X_K, \Theta^{(s)}\right] \quad (76)$$

by generating S realizations from the posterior distribution $p(\Theta|\mathcal{D})$. Similarly, posterior prediction for expected number of arrivals is obtained via

$$E[N_{\tau+t} - N_\tau|\mathcal{D}] = \int E[N_t|X_K, \Theta]p(\Theta|\mathcal{D})d\Theta \quad (77)$$

which we can approximate as a Monte Carlo integral

$$E[N_{\tau+t} - N_\tau|\mathcal{D}] \approx \frac{1}{S} \sum_s E\left[N_t|X_K, \Theta^{(s)}\right] \quad (78)$$

using (74) with realizations from the posterior distribution of Θ. We note that in evaluating the above we compute the posterior distributions for limiting probabilities (51) and the longrun rate (55) using similar Monte Carlo approximations.

As a final remark, note that although we focused on the Bayesian analysis of the distribution of the time of the first arrival and the expected number of arrivals in a given time interval, it is clear that similar analysis can be done in a trivial manner for other quantities discussed in sections 3 and 4.

7 Numerical illustration

We next illustrate the use of our results and the implementation of the Bayesian analysis by using simulated data from a Markov modulated Poisson process (MMPP) where the environmental process consists of three states. Thus, in our notation of section 5, we have the state space $E = \{1, 2, 3\}$ for the Markov process Y. Using the MMPP introduced in section 5, we simulated an arrival process for a period of $\tau = 1{,}595$ units of time during which $K = 99$ changes were observed for the environmental process. It was assumed that the initial state was $X_1 = 1$ and that the generator matrix of the Markov process was given by

$$G = \begin{bmatrix} -0.05 & 0.05 \times 0.65 & 0.05 \times 0.35 \\ 0.10 \times 0.45 & -0.10 & 0.10 \times 0.55 \\ 0.15 \times 0.60 & 0.15 \times 0.40 & -0.15 \end{bmatrix}.$$

The arrival rates $\lambda(1) = 0.2$, $\lambda(2) = 0.3$ and $\lambda(3) = 0.6$ were used in the simulation which provided us with the type of data as in (65).

In our illustration we assume that all the components of Θ are independent and we use diffused prior distributions for each of the components. For the Dirichlet priors in (57), we assume that $\alpha^i_j = 1$ implying that $E[P(i, j)] = 1/3$ for all $i, j \in E$. For the gamma priors of $\lambda(i)$'s, we assume that $a(i) = 1$ and $b(i) = 0.1$ for all $i \in E$; and similarly for the gamma priors on $\mu(i)$'s, we assume that $c(i) = 1$ and $d(i) = 1$ for all $i \in E$. The above choice of parameters implies a high degree of prior uncertainty about the components of Θ. In a situation where prior information exists, the prior parameters can be specified by eliciting best guess values for elements of Θ and uncertainties about these values can be elicited. In what follows, we will focus on posterior analysis of the simulated data and discuss how posterior predictions can be obtained for time to next arrival and for the expected number of arrivals in a given time interval.

The posterior distributions of P_i's can be obtained as Dirichlet distributions given by (68) and the posterior distributions of $\mu(i)$'s and $\lambda(i)$'s are obtained as independent gamma densities as given by (69) and (70), respectively. As previously mentioned, the components of Θ are independent of each other aposteriori. Once these posterior distributions are obtained we can simulate the posterior distribution of the limiting probabilities (51). In Figure 1 we present the density plots for the posterior distributions of $\pi(j)$'s. As implied by Figure 1, the environmental process is expected to be in state 1 most of the time in the longrun. More specifically, the posterior means are given by $E[\pi(1)|\mathcal{D}] = 0.658$, $E[\pi(2)|\mathcal{D}] = 0.216$, and $E[\pi(3)|\mathcal{D}] = 0.125$.

Given the posterior distribution of Θ we can obtain the posterior distribution of the time to the next arrival via simulation using the Monte Carlo approximation (76). To evaluate the Monte Carlo integral for each realization of the posterior distribution of Θ, we need to evaluate the matrix exponential form given by (59) using the eigenvectors of $H(\Theta) = G(\Theta) - \Lambda(\Theta)$. All these evaluations are conditional on the last state, that is, on X_K which corresponds to the state 1 in our case. In Figure 2 we present the posterior predictive distribution of the time to the first arrival and compare it with the actual distribution based on the values of Θ used in simulating the MMPP. We note that the posterior predictive distribution is very close to the actual distribution as shown by the figure.

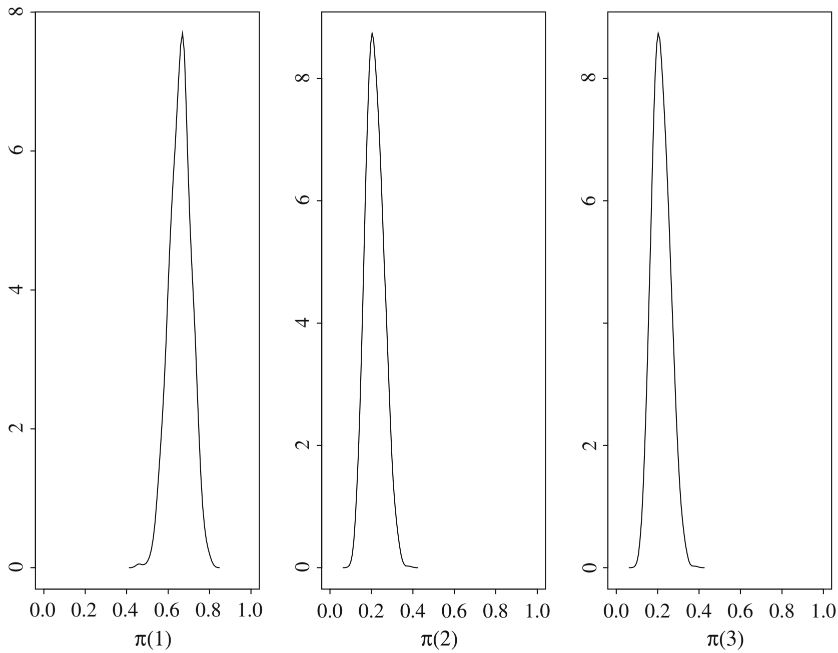

Fig. 1 Posterior distributions of limiting probabilities

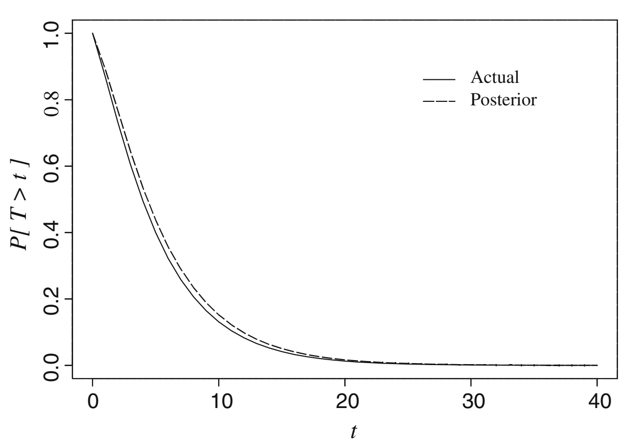

Fig. 2 Comparison of actual and expected posterior distributions of time to the first arrival

The posterior expected number of arrivals for a future interval of $[\tau, \tau + t)$ were obtained for $t = 10, 20, 30, 40$ and 50 using the Monte Carlo integral approximation (78). Computation of this Monte Carlo integral requires evaluation of (74) for each realization of the posterior distribution of Θ. Note that this term involves a matrix exponential form which can be computed as discussed before, and the limiting probabilities and the longrun rate $\hat{\lambda}(\Theta)$ can be evaluated using the corresponding formulas. In Table 1, we present the posterior expected number of arrivals

Table 1 Posterior expected number of arrivals at t=10, 20,30, 40 and 50

t	10	20	30	40	50
$E[N_{\tau+t} - N_\tau \mid \mathcal{D}]$	2.27	4.78	7.38	10.01	12.65
Actual $E[N_{\tau+t} - N_\tau]$	2.37	5.13	8.01	10.94	13.89

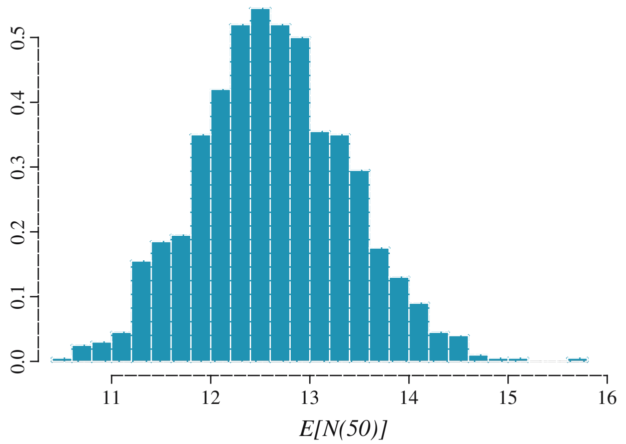

Fig. 3 Posterior distribution of expected number of arrivals at $t = 50$

and compare them with the actual expected number of arrivals based on the values of components of Θ used in simulating the observed data.

We note from the table that the posterior mean arrivals are very close to the actual mean arrivals for shorter time arrivals. The discrepancy is little larger for longer intervals as implied by the cases of $t = 40$ and 50. Such discrepancy is expected since Figure 2 implies longer time between arrivals based on the posterior predictive distribution.

In the Bayesian analysis we can also make probability statements about expected number of arrivals during any interval. This is due to the fact that for given t (74) is a function of Θ and thus we can obtain its probability distribution using samples from the posterior distribution of Θ. This simply requires the evaluation of (74) using the posterior samples. In Figure 3, we present posterior distribution of expected number of arrivals at $t = 50$.

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