



Bayesian portfolio selection with multi-variate random variance models

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Abstract

We consider multi-period portfolio selection problems for a decision maker with a specified utility function when the variance of security returns is described by a discrete time stochastic model. The solution of these problems involves a dynamic programming formulation and backward induction. We present a simulation-based method to solve these problems adopting an approach which replaces the preposterior analysis by a surface fitting based optimization approach. We provide examples to illustrate the implementation of our approach.

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1. Introduction and overview

The portfolio selection problem was originally considered by Markowitz (1952) for an investor who has a sum of money to allocate among K securities. Markowitz's approach to the problem consisted of two steps. First forming a set of efficient portfolios and secondly selecting from the efficient set the one portfolio that provides the

investor with the most suitable combination of risk and return. Extension of the Markowitz model was considered by others where particular objective functions were suggested and mean–variance analysis was introduced (see for example, Farrar, 1962).

An earlier decision theoretic approach to the problem was considered by Mao and Sarndal (1966). The decision theoretic point of view provided a justification for the mean–variance analysis. For example, it can be shown that the formulation of Farrar (1962) follows from maximization of expected utility when security returns

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are normally distributed and the investor's utility function is exponential. Most of the early work in portfolio selection focused on single-period models. Some exceptions to this are the multi-period models of Mossin (1968).

A Bayesian decision theoretic approach to the problem (with revision of uncertainty) was presented in Winkler and Barry (1975) and Barry and Winkler (1976) where Bayesian multi-period models were introduced and computational difficulties were pointed out. More recent Bayesian work includes Quintana (1992) and Quintana and Putnam (1996) where single period adaptive Bayesian models were considered. In other words, these models used sequential updating and one-step ahead portfolio allocations. Polson and Tew (2000) also presented a single period adaptive Bayesian model using a hierarchical model setup. None of these models considered a formal treatment of stochastic volatility. The recent work by Aguilar and West (2000) introduced single period dynamic factor models with stochastic volatility.

Quintana and Putnam (1994) and Quintana et al. (1998) considered the multi-period problem and discussed the difficulties in solving the stochastic dynamic program. In Quintana and Putnam (1994), the authors showed that if the utility function is additive over time, then the optimal solution to the multi-period problem consists of one-period optimal solutions over time. In other words, the one-step look ahead rule is optimal in this case. A comprehensive review of Bayesian work in portfolio management is given in Quintana et al. (2003).

In this paper we will consider the multi-period portfolio selection problem from a Bayesian decision theoretic point of view. In our development we assume that the security returns are described by multi-variate stochastic variance models and in so doing consider both the multi-variate GARCH models of Bollerslev et al. (1988) and state-space type stochastic volatility models of Uhlig (1997) and Liu (2000). We note that the presented approach is general and can be applied to other types of multi-variate stochastic volatility models considered by Harvey et al. (1994) and Soyer and Tanyeri (2003). Our formulation of

the problem will follow a setup similar to that considered in Winkler and Barry (1975). We present a solution to the decision problem based on Monte Carlo methods that alleviate some of the difficulties. In Section 2 we present the decision theoretic setup for the portfolio selection problem. We focus on the two-period problem with stationary returns and show the difficulties involved in the preposterior analysis. In Section 3 we consider the extension of the problem where the returns following two different classes of multi-variate stochastic variance models. We introduce a standard Monte Carlo approach to solve the two-period portfolio selection problem in the multi-variate stochastic variance models in Section 4 and illustrate the computational burden and inefficiencies involved in implementation of the standard Monte Carlo method. To alleviate these problems we present an alternative Monte Carlo method. Our method is based on the curve/surface fitting approach of Muller and Parmigiani (1995) that was developed for optimal design problems. This approach enables us to perform the preposterior analysis in an efficient manner. An implementation of this approach is illustrated in Section 5 using the two stochastic variance models and the results are compared with the standard Monte Carlo method.

We note that the multi-period allocation problem that we consider here arises also in derivatives pricing in the context of discrete time hedging problem. As pointed out by one of the referees, these problems typically involve allocation between two assets and thus are well-suited to the proposed method.

2. Decision theoretic setup for the portfolio selection problem

Let W_t denote the wealth of the investor at the end of time period t and W_0 denote the initial wealth of the investor at time 0. Also, we define r_t^k as the return from security k during time period t . If we assume that there are no transaction costs and the portfolio will be chosen from K securities, then the investor's wealth at the end of period t is given by

$$W_t = \sum_{k=1}^K W_t^k, \tag{2.1}$$

where W_t^k is the amount invested in security k at the beginning of time period $t + 1$. The wealth of the investor at the end of time period $(t + 1)$ can be written as

$$W_{t+1} = \sum_{k=1}^K (1 + r_{t+1}^k) W_t^k = W_t + \sum_{k=1}^K r_{t+1}^k W_t^k. \tag{2.2}$$

Following the development of Winkler and Barry (1975), we assume that the investor’s objective is to maximize the utility of wealth at the end of a finite time period T , that is, to maximize $U(W_T)$. The form of the utility function describes the investor’s attitude towards risk and provides the investor with a combination of risk and return that reflects this attitude. As pointed out by Mao and Sarndal (1966), in finance literature various suggestions have been made to choose a portfolio and these strategies can be justified by specifying different utility functions. For example, the mean–variance analysis is justified by a quadratic utility function irrespective of the distribution of security returns. Another form of mean–variance analysis, as in Farrar (1962), is justified by an exponential utility function when security returns are normally distributed.

In the multi-period problem with a finite horizon T , the investor will be maximizing the utility $U(W_T)$ by sequentially choosing the decision variables $W_t^k, t = 0, \dots, T - 1$ and $k = 1, \dots, K$, at different points in time based on the available information. That is, the optimal allocation will be revised as the random quantities $r_t^k, t = 1, \dots, T$, are observed over time. The multi-period optimal allocation problem can be represented as a sequential decision problem as given by the T -stage decision tree in Fig. 1.

In Fig. 1 the decision nodes at time periods $t = 0, \dots, T - 1$ are denoted by D_0, D_1, \dots, D_{T-1} and the random (observation) nodes are denoted by O_1, O_2, \dots, O_T . At $t = 0$, given the initial wealth W_0 , at decision node D_0 , the investor will determine $W_0^K = (W_0^1, \dots, W_0^K)$, that is, the amounts invested in the K securities. At time period $t = 1$, the returns from K securities, $r_1 = (r_1^1, r_1^2, \dots, r_1^K)$ are observed. This is denoted by the random node O_1 in the decision tree. Next, given the observed returns r_1 , at decision node D_1 the investor will determine $W_1^K = (W_1^1, \dots, W_1^K)$ and this process is repeated at the subsequent nodes. As pointed out by Winkler and Barry (1975), the solution of this problem involves a dynamic programming formulation and backward induction. Solution of the tree proceeds in the usual way by taking expectation at random nodes and maximizing the expected utility at the decision nodes (see Lindley, 1985). At decision node $(T - 1)$, given the observed returns $r_s^k, s = 1, \dots, T - 1$ and $k = 1, \dots, K$, the decision variables $W_{T-1}^K = (W_{T-1}^1, W_{T-1}^2, \dots, W_{T-1}^K)$ are chosen such that

$$\max_{W_{T-1}^K} E_{T-1}[U(W_T)], \tag{2.3}$$

where E_{T-1} denotes the expectation conditional on the returns from the first $(T - 1)$ periods. In other words, the expectation E_{T-1} is taken with respect to the probability distribution of $r_T = (r_T^1, r_T^2, \dots, r_T^K)$ given $\mathbf{D}_{T-1} = (r_1, r_2, \dots, r_{T-1})$. We will denote this distribution by $p(r_T | \mathbf{D}_{T-1})$ assuming all of its parameters are known.

Let $U_{T-1}^*(W_T)$ denote the optimal value of the expected utility corresponding to the optimal allocation $W_{T-1}^{K*} = (W_{T-1}^{1*}, W_{T-1}^{2*}, \dots, W_{T-1}^{K*})$ at time $(T - 1)$. At time $(T - 2)$, that is, at decision node $(T - 2)$, the optimal allocation is obtained via

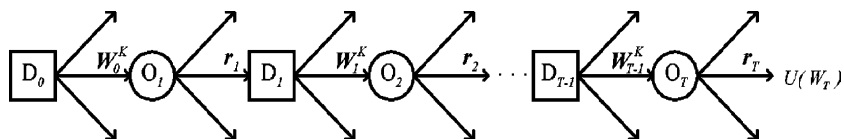


Fig. 1. Decision tree for the multi-period portfolio selection problem.

$$\max_{W_{T-2}^K} E_{T-2}[U_{T-1}^*(W_T)], \tag{2.4}$$

where $U_{T-1}^*(W_T) = E_{T-1}[U(W_{T-1} + \sum_{k=1}^K r_T^k W_{T-1}^k)]$. In general at time period t , given the observed returns \mathbf{D}_t the decision variables $W_t^K = (W_t^1, W_t^2, \dots, W_t^K)'$ are chosen such that

$$\max_{W_t^K} E_t[U_{t+1}^*(W_T)], \tag{2.5}$$

where E_t denotes the expectation conditional on the returns $r^{(t)}$ from the first t periods and $U_{t+1}^*(W_T)$ is the expected utility corresponding to the optimal allocation at time periods $t + 1, \dots, T - 1$. In this case expectation, E_t , is taken with respect to the probability distribution $p(r_{t+1}|\mathbf{D}_t)$.

Continuing in this manner, at time 0, the optimal allocation for investment period 1, is obtained by solving

$$\max_{W_0^K} E_0 \left[\max_{W_1^K} E_1 \left[\dots \max_{W_{T-1}^K} E_{T-1}[U(W_T)] \dots \right] \right]. \tag{2.6}$$

Note that (2.6) is equivalent to solving

$$\max_{W_0^K} E_0[U_1^*(W_T)], \tag{2.7}$$

and for the special case of a single period problem, that is, $T = 1$, the above reduces to

$$\max_{W_0^K} E_0[U(W_1)]. \tag{2.8}$$

In general, even for the case of a finite horizon problem, calculation of the optimal allocation is not trivial as it involves implicit computation of expectations and maximizations at each time period. Also, the Bayesian strategy requires revision of uncertainty about the parameters of the distribution of r_t after each time period. Thus, the evaluation of the expectations may not be analytically tractable. If $T = 1$, then we have a single period problem implying that the investor follows a myopic policy which requires only looking at one period ahead. In this case solution can be obtained for some simple cases.

In what follows we will consider a two-period problem with a specific form for the utility function as well as for the probability distribution of the returns.

2.1. A two-period problem

To illustrate an implementation of the Bayesian decision theoretic setup now we consider a two-period portfolio selection problem where the investor needs to determine the optimal portfolio so that the utility of wealth at the end of period 2, that is,

$$U(W_2) = U\left(W_1 + \sum_{k=1}^K r_2^k W_1^k\right) = U(W_1 + r_2' W_1^K) \tag{2.9}$$

is maximized. The decision tree associated with the two-period portfolio selection problem is shown in Fig. 2.

We assume that at time period t the K dimensional return vector r_t is normally distributed with mean vector μ and covariance matrix Σ . We will denote this distribution as

$$(r_t|\mu, \Sigma) \sim N(\mu, \Sigma), \tag{2.10}$$

and following Winkler and Barry (1975) we will assume that r_t 's are independent random vectors. For our development in this section we also assume that μ is unknown and uncertainty about μ is described by a normal prior

$$(\mu|m_0, C_0) \sim N(m_0, C_0), \tag{2.11}$$

where m_0 and C_0 are specified quantities. The covariance matrix Σ is assumed to be known in this section.

The two-period decision problem shown in Fig. 2 is solved in the conventional manner by *folding back* the decision tree via taking expectations at the random nodes and maximizing the expected utility at the decision nodes. We assume that the investor has an exponential utility function

$$U(W_2) = -\exp(-AW_2), \tag{2.12}$$

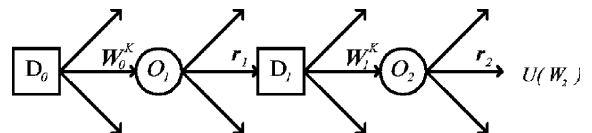


Fig. 2. Decision tree for the two-period portfolio selection problem.

where $A > 0$ and $1/A$ is referred to as the “risk tolerance” and describes the degree of risk aversion. Smaller the A is the higher the risk tolerance is for the investor.

At the random node O_2 the expected utility $E_1[U(W_2)]$ is evaluated by using the probability distribution of r_2 given r_1 , that is by using $p(r_2|r_1)$. Given the above setup, via standard Bayesian updating it can be shown that

$$(r_2|r_1) \sim N(m_1, C_1 + \Sigma), \tag{2.13}$$

where

$$m_1 = C_1^{-1}(C_0^{-1}m_0 + \Sigma^{-1}r_1) \quad \text{and} \\ C_1 = (C_0^{-1} + \Sigma^{-1})^{-1}. \tag{2.14}$$

Since $W_2 = W_1 + r_2'W_1^K$, given r_1 , W_2 will also have a normal distribution. Thus, at random node O_2 the expected utility $E_1[U(W_2)]$ is obtained as

$$E_1[U(W_2)] = - \int \exp(-AW_2)p(W_2|r_1)dW_2. \tag{2.15}$$

Using the properties of normal distribution it can be shown that

$$E_1[U(W_2)] = - \exp \left\{ -A \left[(W_1 + m_1'W_1^K) - \frac{A}{2} ((W_1^K)'P_1W_1^K) \right] \right\}, \tag{2.16}$$

where $P_1 = C_1 + \Sigma$. The optimal portfolio for period two is obtained as

$$W_1^{K*} = \frac{P_1^{-1}m_1}{A} \tag{2.17}$$

by maximizing (2.16) with respect to W_1^K at decision node D_1 . Optimal value of the expected utility, $U_1^*(W_2)$, is obtained by evaluating (2.16) using $W_1^K = W_1^{K*}$. Next at random node O_1 , $E_0[U_1^*(W_2)]$ needs to be obtained using the predictive distribution $p(r_1)$ which is a normal density with mean vector m_0 and covariance matrix $(C_0 + \Sigma)$. Note that $U_1^*(W_2)$ involves m_1 and W_1 which both are functions of r_1 and thus the expectation $E_0[U_1^*(W_2)]$ cannot be evaluated in closed form and the optimal portfolio cannot be obtained analytically. In this case one alternative approach is to use a Monte Carlo method to evaluate the

expectation and then find the portfolio. But this is not easy to implement for problems with $T > 2$ periods and may become computationally inefficient in the multi-variate random variance models that are considered in Section 3.

3. Multi-variate random variance models

In this section we assume that the conditional variance of returns is a function of time, that is,

$$(r_t|\mu, \Sigma_t, D_{t-1}) \sim N(\mu, \Sigma_t), \tag{3.1}$$

where $D_{t-1} = (r_1, r_2, \dots, r_{t-1})$ and we treat Σ_t as a stochastic process. In so doing, we will consider two classes of models. The first model is a state-space type stochastic volatility model which has been considered by Uhlig (1997) and Liu (2000) and the second model is the multi-variate GARCH model of Bollerslev et al. (1988). As previously mentioned other types of stochastic processes such as discrete time gamma processes can also be considered to model conditional variances. In what follows we first introduce the models and then consider the portfolio selection problem using these models and discuss computational issues.

3.1. State space volatility model

We first rewrite (3.1) as a linear model with observation noise ξ_t as

$$r_t = \mu_t + \xi_t, \tag{3.2}$$

where $\xi_t|D_{t-1} \sim N(0, \Phi_t^{-1})$ and $\Phi_t = \Sigma_t^{-1}$ is the $K \times K$ precision matrix. We will assume that the prior for mean return vector μ_t is given by

$$\mu_t|\Phi_t, D_{t-1} \sim N(m_{t-1}, \Phi_t^{-1}/v), \tag{3.3}$$

where v is a specified precision parameter and m_{t-1} is the known mean vector at time $t - 1$. Conditional on Φ_t 's, μ_t 's are independent random quantities in (3.3). Following Uhlig (1997) and Liu (2000) the precision matrix Φ_t is assumed to follow a first-order Markov structure as

$$\Phi_t = C(\Phi_{t-1})' \beta_t C(\Phi_{t-1})/\delta, \tag{3.4}$$

where $C(\Phi_{t-1})$ is the upper Cholesky decomposition of Φ_{t-1} . Given D_{t-1} we assume that β_t will

have a matrix beta distribution (see Uhlig, 1994 for definition) denoted as

$$\beta_t | \mathbf{D}_{t-1} \sim M\text{Beta}(\delta n_{t-1}/2, (1 - \delta)n_{t-1}/2), \quad (3.5)$$

and that Φ_{t-1} has a Wishart distribution (see Uhlig, 1994) with scale matrix \mathbf{P}_{t-1} and degrees of freedom n_{t-1} denoted as

$$\Phi_{t-1} | \mathbf{D}_{t-1} \sim \text{Wish}(\mathbf{P}_{t-1}, n_{t-1}). \quad (3.6)$$

We note that in the above $0 < \delta < 1$ where δ is typically larger than 0.9 and $\delta n_{t-1} > K$ for the matrices to be positive definite. It can be shown that (3.4)–(3.6) imply a prior for Φ_t as

$$\Phi_t | \mathbf{D}_{t-1} \sim \text{Wish}(\delta \mathbf{P}_{t-1}, \delta n_{t-1}). \quad (3.7)$$

The above development yields a random walk type of evolution for the precision matrix from one period to the next. We note that (3.6) and (3.7) imply that

$$E(\Phi_{t-1} | \mathbf{D}_{t-1}) = E(\Phi_t | \mathbf{D}_{t-1}) = \mathbf{P}_{t-1}^{-1} n_{t-1}, \quad (3.8)$$

that is, the means are the same, but the dispersion increases as we move from period $(t - 1)$ to t . More specifically, \mathbf{P}_{t-1} , the scale matrix of Φ_{t-1} , changes to $\delta \mathbf{P}_{t-1}$ in (3.7).

An attractive feature of the above model is that if we start with a prior $\Phi_0 | \mathbf{D}_0 \sim \text{Wish}(\mathbf{P}_0, n_0)$ at time 0, then a conjugate Bayesian analysis is available. It can be shown that

$$\Phi_t | \mathbf{D}_t \sim \text{Wish}(\mathbf{P}_t, n_t), \quad (3.9)$$

and

$$\mu_t | \Phi_t, \mathbf{D}_t \sim N(\mathbf{m}_t, \Phi_t^{-1} / (1 + v)), \quad (3.10)$$

where $n_t = \delta n_{t-1} + 1$,

$$\mathbf{P}_t = \delta \mathbf{P}_{t-1} + \frac{v}{1+v} (\mathbf{r}_t - \mathbf{m}_{t-1})' (\mathbf{r}_t - \mathbf{m}_{t-1}),$$

and

$$\mathbf{m}_t = \frac{1}{1+v} \mathbf{r}_t + \frac{v}{1+v} \mathbf{m}_{t-1}.$$

3.2. Multi-variate GARCH models

A multi-variate extension of the univariate ARCH and generalized ARCH models of Engle (1982) and Bollerslev (1986), is introduced by Bollerslev et al. (1988). Using the observation model

$$\mathbf{r}_t = \boldsymbol{\mu} + \boldsymbol{\xi}_t, \quad (3.11)$$

where $\boldsymbol{\xi}_t | \mathbf{D}_{t-1} \sim N(0, \boldsymbol{\Sigma}_t)$, the multi-variate GARCH(q, p) model is defined as

$$\begin{aligned} \text{vech}(\boldsymbol{\Sigma}_t) = & \mathbf{A}_0 + \sum_{i=1}^q \mathbf{A}_i \text{vech}(\boldsymbol{\xi}_{t-i} \boldsymbol{\xi}'_{t-i}) \\ & + \sum_{j=1}^p \mathbf{B}_j \text{vech}(\boldsymbol{\Sigma}_{t-j}). \end{aligned} \quad (3.12)$$

In (3.12) $\text{vech}(\cdot)$ denotes the column stacking operator of the lower portion of a symmetric matrix, \mathbf{A}_0 is a $s = 1/2 \times K(K + 1)$ vector, \mathbf{A}_i 's and \mathbf{B}_j are $s \times s$ matrices. We note that certain conditions need to be satisfied in (3.12) for $\boldsymbol{\Sigma}_t$ to be a positive definite matrix. In what follows, we will consider the ARCH models in (3.12), that is, the case where $p = 0$. Similar to the univariate ARCH models, multi-variate ARCH(q) models imply that $\text{vech}(\boldsymbol{\xi}_t \boldsymbol{\xi}'_t)$ follows a vector autoregressive process of order q . In other words, we can motivate an ARCH(q) process via

$$\text{vech}(\boldsymbol{\xi}_t \boldsymbol{\xi}'_t) = \mathbf{A}_0 + \sum_{i=1}^q \mathbf{A}_i \text{vech}(\boldsymbol{\xi}_{t-i} \boldsymbol{\xi}'_{t-i}) + \mathbf{u}_t, \quad (3.13)$$

where $\mathbf{u}_t = \text{vech}(\boldsymbol{\xi}_t \boldsymbol{\xi}'_t) - \text{vech}(\boldsymbol{\Sigma}_t)$ is a zero-mean white-noise process.

We note that the model (3.12) is highly parameterized. For example, for $K = 2$, the first order ARCH model can be written as

$$\begin{pmatrix} \Sigma_{11t} \\ \Sigma_{12t} \\ \Sigma_{22t} \end{pmatrix} = \begin{bmatrix} \alpha_{01} \\ \alpha_{02} \\ \alpha_{03} \end{bmatrix} + \begin{bmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{bmatrix} \begin{bmatrix} \xi_{1,t-1}^2 \\ \xi_{1,t-1} \cdot \xi_{2,t-1} \\ \xi_{2,t-1}^2 \end{bmatrix}. \quad (3.14)$$

A natural simplification is obtained by assuming a diagonal structure for the \mathbf{A}_1 matrix in (3.14), that is, $\alpha_{ij} = 0$ if $i \neq j$. As pointed out by Bollerslev et al. (1988), this simply implies that each covariance depends only on its past values, that is,

$$\begin{pmatrix} \Sigma_{11t} \\ \Sigma_{12t} \\ \Sigma_{22t} \end{pmatrix} = \begin{bmatrix} \alpha_{01} + \alpha_{11} \xi_{1,t-1}^2 \\ \alpha_{02} + \alpha_{22} \xi_{1,t-1} \cdot \xi_{2,t-1} \\ \alpha_{03} + \alpha_{33} \xi_{2,t-1}^2 \end{bmatrix}. \quad (3.15)$$

Similar simplifications can be obtained for multi-variate GARCH models which are motivated by $\text{vech}(\xi, \xi')$ being a vector ARMA process.

A Bayesian analysis of the multi-variate GARCH model using any form of prior distributions for μ , A_0 , A_i 's and B_j 's, requires use of Markov chain Monte Carlo (MCMC) methods. The MCMC approach introduced by Muller and Pole (1998) for univariate GARCH models has been extended to multi-variate GARCH models in Soyer and Tanyeri (2004).

3.3. Portfolio selection using random variance models

The two-period portfolio selection problem presented in Section 2.1 requires evaluation of expected utility at random nodes O_1 and O_2 . As we have pointed out earlier, the expectation at node O_2 is with respect to the posterior predictive distribution $p(r_2|r_1)$. In Section 2.1 where this distribution was given as a normal density in (2.13), we were able to evaluate the expected utility in (2.15).

In the state space volatility model, based on the posterior results (3.9) and (3.10) it is again possible to obtain the posterior predictive distribution $p(r_2|r_1)$. In this case $p(r_2|r_1)$ is a multi-variate- t density with degrees of freedom $(\delta n_1 - K + 1)$, mean vector m_1 , and scale matrix P_1 . This also implies that W_2 given r_1 has a multi-variate- t distribution and as a result the expectation in (2.15) cannot be evaluated in closed form and thus the optimal portfolio at decision node D_1 cannot be obtained analytically.

In the multi-variate GARCH models, the posterior analysis cannot be obtained analytically and as a result the exact form of the posterior predictive distribution $p(r_2|r_1)$ is not known. Using MCMC methods we can draw samples from the posterior predictive distribution, but this does not allow us to solve the expectation in (2.15) analytically. Thus, in both of these cases solution of the optimal portfolio selection problem requires use of Monte Carlo methods. In the sequel, we will present Monte Carlo-based methods to solve the multi-period portfolio selection problem with random variances.

4. Monte Carlo based approaches for portfolio selection problem

We consider the two-period problem presented in Section 2.1 with the conditional variance is described by the random variance models of Section 3 and illustrate how the decision problem can be solved using Monte Carlo based methods. In so doing, we first present a standard Monte Carlo approach and illustrate how this method becomes computationally inefficient in Section 4.1. To alleviate the computational problems, in Section 4.2 we introduce an alternate Monte Carlo method to solve the two-period problem by adopting the simulation based approach of Muller and Parmigiani (1995) where the preposterior analysis is replaced by a surface fitting approach. We discuss extension of the proposed approach to multi-period portfolio selection problems in general as well as the related computational issues.

4.1. Standard Monte Carlo approach

Let Θ_t denote unknown parameters of the model for the returns at time t which can be either described by the state space model of Section 3.1 or by the multi-variate GARCH model of Section 3.2. The standard Monte Carlo solution of the decision tree given in Fig. 2 involves Monte Carlo averages at random nodes O_1 and O_2 and maximization of these Monte Carlo averages at random nodes D_0 and D_1 . Folding back the decision tree using the Monte Carlo approach consists of the following steps:

- Step 1: Choose initial wealth W_0 and portfolio W_0^K . Also, choose the vector of weights $w_1^{js} = (w_1^{1s}, w_1^{2s}, \dots, w_1^{Ks})$, for $j = 1, \dots, J$, where w_1^k represents the portion of the wealth W_1 which will be allocated to security k in period 2's portfolio. Note that $\sum_{k=1}^K w_1^k = 1$.
- Step 2: Generate Θ_1^i and r_1^i , $i = 1, \dots, I$, based on the prior $p(\Theta_1)$ and prior predictive distribution $p(r_1|\Theta_1)$ for the model. Compute $W_1^i = W_0 + r_1^i W_0^K$ and for each $i = 1, \dots, I$, using the weight vector w_1^j ,

obtain the portfolios $W_1^{Kj}(i) = w_1^j W_1^i$ for $j = 1, \dots, J$.

Step 3: For each Θ_1^i and r_1^i , generate Θ_2^l and r_2^l , $l = 1, \dots, L$, based on the model. Compute the utility $U(W_2^l)$ where $W_2^l = W_1^j + r_2^l W_1^{Kj}(i)$ for $l = 1, \dots, L$, $j = 1, \dots, J$ and $i = 1, \dots, I$.

Step 4: At random node O_2 evaluate the expectation using the Monte Carlo average

$$E[U(W_2) | W_0^K, r_1^i, W_1^{Kj}(i)] = \frac{1}{L} \sum_{l=1}^L U(W_2^l), \tag{4.1}$$

for $l = 1, \dots, L$, $j = 1, \dots, J$ and $i = 1, \dots, I$.

Step 5: At decision node D_1 optimize (4.1) with respect to $W_1^K(i)$, that is, find $W_1^{K*}(i)$ that maximizes

$$E[U(W_2) | W_0^K, r_1^i, W_1^{Kj}(i)], \quad j = 1, \dots, J, \text{ and obtain}$$

$$U_1^*(W_2) = E[U(W_2) | W_0^K, r_1^i, W_1^{K*}(i)], \tag{4.2}$$

where $W_1^{K*}(i)$ is a function of r_1^i .

Step 6: At random node O_1 compute the preposterior expected utility using the Monte Carlo average

$$E[U_1^*(W_2) | W_0^K] = \frac{1}{I} \sum_{i=1}^I E[U(W_2) | W_0^K, r_1^i, W_1^{K*}(i)]. \tag{4.3}$$

Step 7: Go to Step 1 and repeat steps 2–6 for a different W_0^K and finally at decision node D_0 find the maximizer of $E[U_1^*(W_2) | W_0^K]$ which gives us the optimal first period portfolio.

For implementation of the Monte Carlo approach, in the state space model of Section 3.1 we have $\Theta_t = (\mu_t, \Phi_t)$ and in the multi-variate GARCH model $\Theta_t = (\mu, \Sigma_t)$ where Σ_t is a function of A_0, A_j 's and B_j 's as defined in (3.12). In the state space model initially the prior of Φ_0 is specified and Φ_1 is generated using (3.7) and given Φ_1 with specified value of m_0 the mean vector μ_1 is generated using (3.3) and the return vector r_1 is generated

from (3.2). In step 3 given Φ_1, Φ_2 is generated using the transition model (3.4) with β_2 is drawn from (3.5) for Given Φ_2 and r_1, μ_2 is drawn from (3.3) with m_1 is updated accordingly. In the multi-variate GARCH model, if we assume that we have a multi-variate GARCH(1,1) model then the prior $p(\mu, A_0, A_1, B_1)$ is specified. In so doing, initially we can choose $\text{vech}(\Sigma_0) = A_0$ and assume that $\xi_0 = 0$ where 0 is a K dimensional vector of zeros. Note that in choosing the prior of A_0, A_1, B_1 we need to ensure the positive definiteness of the Σ_1 matrix. Once μ, A_0, A_1 and B_1 are generated, then Σ_1 is drawn from (3.12) Next the error ξ_1 is drawn and the return vector r_1 is generated from (3.11). Given Σ_1 and ξ_1 , for the second period, Σ_2 is drawn from (3.12) and r_2 is drawn from (3.11).

Note that even for the two-period problem, the implementation of the above Monte Carlo approach is not computationally efficient. The approach requires, for each first period portfolio W_0^K , specification of J second period portfolios W_1^K and $I \times L$ draws from the returns for each portfolio combination. Thus, evaluation of the Monte Carlo integral (4.1) requires a large computational effort in obtaining $U(W_2)$ for each W_0^K .

4.2. Surface fitting approach

The computational burden involved in the use of standard Monte Carlo approach is common to many Bayesian decision problems such as the optimal design problems. As pointed out by Muller (1999), such a large scale simulation for each value of the decision variable fails to exploit the continuity of the expected utility surface. To avoid the computational burden in use of the standard Monte Carlo approach, we will adopt the curve/surface fitting method proposed by Muller and Parmigiani (1995) to the portfolio selection problem. This approach facilitates preposterior analysis by replacing the integration steps required at random nodes of the decision tree by smoothing steps. The maximization steps at the decision nodes are replaced by the maximization of the fitted smooth surfaces.

The proposed approach consists of the following steps:

- Step 1:* Choose initial wealth W_0 and portfolio W_0^{Ks} . Also, choose the vector of weights $w_1^s = (w_1^{1s}, w_1^{2s}, \dots, w_1^{Ks})$, for $s = 1, \dots, S$, where $\sum_{k=1}^K w_1^k = 1$.
- Step 2:* Generate $(\Theta_1^s$ and $r_1^s)$, $s = 1, \dots, S$, using $p(\Theta_1)$ and $p(r_1|\Theta_1)$ for the model. Compute $W_1^s = W_0 + r_1^s W_0^{Ks}$ and obtain the portfolios $W_1^{Ks} = w_1^s W_1^s$.
- Step 3:* Given $(\Theta_1^s$ and $r_1^s)$, generate Θ_2^s and r_2^s , $s = 1, \dots, S$, based on the model. Using S points evaluate the utility $U_2^s = U(W_2^s)$, with $W_2^s = W_1^s + r_2^s W_1^{Ks}$ and record the Monte Carlo sample points $(W_0^{Ks}, W_1^{Ks}, r_1^s, r_2^s)$ and U_2^s .
- Step 4:* Fit a surface $L_2(W_0^{Ks}, W_1^{Ks}, r_1^s, r_2^s)$ to the points $(W_0^{Ks}, W_1^{Ks}, r_1^s, U_2^s)$. Note that this step is analogous to taking expectation at random node O_2 of the decision tree.
- Step 5:* Find the maximum of fitted surface $L_2(W_0^{Ks}, W_1^{Ks}, r_1^s, r_2^s)$ over W_1^k and replace the Monte Carlo points by (W_0^{Ks}, r_1^s, u_2^s) where $u_2^s = L_2(W_0^{Ks}, W_1^{K*}, r_1^s)$ and W_1^{K*} is the maxima of $L_2(W_0^{Ks}, W_1^{Ks}, r_1^s)$. Here W_1^{K*} is the solution of the portfolio selection problem for the second period and it depends on (W_0^{Ks}, r_1^s) . This is analogous to the maximization step at decision node D_1 of the decision tree.
- Step 6:* Fit a surface $L_1(W_0^{Ks})$ to the points (W_0^{Ks}, u_2^s) . This is analogous to the expectation taken at random node O_1 of the tree.
- Step 7:* Find the maximum of the fitted surface $L_1(W_0^{Ks})$ over W_0^k . This is analogous to the maximization step at decision node D_0 of the decision tree and the maxima W_0^{K*} is the solution of the portfolio selection problem for the first period.

For the multi-period portfolio selection problems with $T > 2$ periods, steps 3 through 5 of the proposed algorithm would be repeated $(T - 1)$ times.

In implementation of the surface fitting algorithm, generation of random quantities Θ_1 , r_1 , Θ_2 , and r_2 under both models follow along the same lines presented for the standard Monte Carlo approach in Section 4.1. However, the computa-

tional effort that is required is a lot more modest for the surface fitting approach. As pointed out by Muller (1999), the surface fitting approach does not use a large scale simulation for each allocation but instead exploits the continuity of the expected utility surface. In Step 4 of the algorithm a non-parametric surface $L_2(W_0^{Ks}, W_1^{Ks}, r_1^s)$ is fitted to the points $(W_0^{Ks}, W_1^{Ks}, r_1^s, U_2^s)$. We can think about the points $(W_0^{Ks}, W_1^{Ks}, r_1^s, U_2^s)$, $s = 1, \dots, S$, as the data of a regression problem with U_2 as the dependent variable and (W_0^k, W_1^k, r_1) as the vector of predictors, or the independent variables. Thus, this step is an approximation to the conditional expectation with respect to r_2 . In Step 5, we maximize the fitted surface L_2 with respect to W_1^k , which is a multi-dimensional optimization. This usually requires numerical techniques since most nonparametric regression surface estimators are not available in closed forms. In Step 6 surface $L_1(W_0^{Ks})$ is fitted to the points (W_0^{Ks}, u_2^s) and in Step 7 $L_1(W_0^k)$ is maximized with respect to W_0^k which is a K -dimensional optimization. The optimal solution at the decision node D_0 of the tree gives us the optimal portfolio for the first period. As previously mentioned, it is important to note that the optimal decision at D_1 represents the optimal portfolio for the second period and this decision depends on both the first period portfolio and r_1 , the returns outcome of the first period.

The choice of the smoothing technique used in steps 4 and 6 depend on the dimensionality of the problem. Nonparametric regression methods or commonly used smoothing methods can be appropriate. In our implementations in Section 5, we used the local regression model *Loess* of Chambers and Hastie (1992). In some applications the fitted expected utility surfaces may have tendency to be flat, but it is possible to *tighten* the expected utility surfaces using power transformations as suggested by Muller (1999).

5. Illustrations of the Monte Carlo approach

We will illustrate implementations of the surface fitting approach using the two multi-variate random variance models of Section 3. In so doing, we will consider a two-period portfolio selection

problem as in Fig. 2 and we assume that there are $K = 3$ securities in the portfolio where the third security is a risk-free asset. For illustrative purposes we assume that we do not have any historical data and therefore all the simulations will be based on the prior and prior predictive distributions.

In our illustrations we assume that the initial wealth is $W_0 = 1$ and the return on riskless asset is 0.02. The risk tolerance parameter is specified as $A = 0.9$ in (2.12). We will use the surface fitting approach based on $S = 1000$ simulations and portfolios. To see the accuracy of our results we will also solve the problem using a standard Monte Carlo approach based on a large scale simulation of 1000^4 samples.

For the state space model of Section 3.1, we will specify the discount parameter as $\delta = 0.9$ and the parameters of the Wishart prior for $\Phi_0 | \mathbf{D}_0$ as $n_0 = 1000$ and

$$P_0 = \begin{bmatrix} 0.020 & 0.002 \\ 0.002 & 0.010 \end{bmatrix}.$$

The prior parameters of μ_0 are chosen as $v = 0.8$ and $m_0 = [0.05 \ 0.05]'$. Note that the choice of these prior parameters ensures the positive definiteness of the precision matrices.

We use a multi-variate ARCH(1) model with a diagonal structure given by (3.14) in our illustration. In so doing, we assume that the components of A_0 vector are independent gamma distributed random quantities, that is, $\alpha_{0,i} \sim \text{Gamma}(0.75, 1)$, $i = 1, \dots, 3$. Similarly, the diagonal matrix A_1 has independent gamma components, $\alpha_{1,i,i} \sim \text{Gamma}(0.75, 1)$, $i = 1, \dots, 3$. The mean vector μ is assumed to have a normal prior with mean vector $m_0 = [0.03 \ 0.025]'$ and variance covariance matrix $C_0 = \begin{bmatrix} 0.010 & 0.006 \\ 0.006 & 0.010 \end{bmatrix}$. Furthermore, we assume that a priori A_0 , A_1 , and μ are independent of each other and specify $\xi_0 = [0.0 \ 0.0]'$.

Note that since the initial wealth is $W_0 = 1$, we can think of the initial portfolio as the weight vector $W_0^3 = (w_0^1, w_0^2, w_0^3)$ such that $\sum_{k=1}^3 w_0^k = 1$. Thus, w_0^3 represents proportion of W_0 which is allocated to the risk-free asset in the first period's portfolio.

We first present results of implementing the surface fitting algorithm of Section 4.2 with the state space volatility model using a simulation size of $S = 1000$ in the algorithm. Based on the above priors the optimal portfolio for period 1 is chosen as $w_0^{1*} = 0.352$, $w_0^{2*} = 0.598$ implying that $w_0^{3*} = 0.05$. In other words, approximately 35% of the wealth is allocated to the first security, 60% is allocated to the second security and remaining 5% is allocated to the risk-free asset. To obtain the exact optimal portfolio we used the standard Monte Carlo method of Section 4.1 and ran a large scale simulation of 1000^4 samples. This also resulted in an allocation of $w_0^{1*} = 0.35$, and $w_0^{2*} = 0.60$ implying the accuracy of the results from surface fitting algorithm. The optimal first period portfolio weights for the surface fitting and the standard Monte Carlo approaches are shown in Table 1.

In Fig. 3, we present the plot of the fitted expected utility surface that we obtain at the random node O_1 of the decision tree. In other words, this corresponds to the fitted surface $L_1(W_0^2)$ to the points (W_0^2, u_2) in Step 6 of the algorithm presented in Section 4.2. Note that the optimal portfolio $w_0^{1*} = 0.352$ and $w_0^{2*} = 0.598$ is obtained by maximizing this expected utility surface with re-

Table 1
Comparison of optimal portfolios using the surface fitting and standard Monte Carlo (SMC) approaches

w_0^{1*}	w_0^{2*}	w_0^{1*} (SMC)	w_0^{2*} (SMC)
0.352	0.598	0.350	0.600

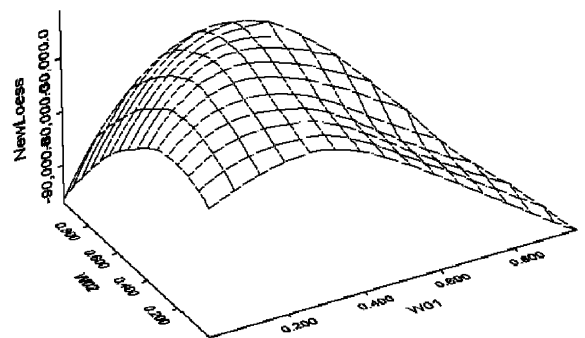


Fig. 3. Expected utility surface as a function of w_0^1 and w_0^2 .

spect to w_0^1 and w_0^2 at the decision node D_0 as implied by the Step 7 of the algorithm.

As we have previously pointed out, the selection of the optimal portfolio for the second period will be made after observing the first period returns. Thus, this selection is a function of both W_0^3 , the portfolio of the first period, and the observed return vector r_1 . Two examples of optimal portfolios W_1^{3*} for period two for selected values of W_0^3 and r_1 are shown in Table 2. Columns 1 and 2 of the table show the selected allocations to the risky securities whereas columns 3 and 4 show the returns observed at period 1. Optimal portfolio allocations for period 2 using the surface fitting approach are illustrated in columns 5 and 6 and the optimal portfolios using the standard Monte Carlo approach are given in columns 7 and 8.

When we compare the results of the two approaches, we see that they are very close. Thus, the surface fitting algorithm, which is based on a simulation effort of only $S = 1000$ samples, seems very accurate. Note that under the first case in the table, a portfolio of $w_0^1 = 0.712$ and $w_0^2 = 0.007$ is specified implying that $w_0^3 = 0.281$ is allocated to the risk-free asset. During the first period the returns on risky securities are observed as $r_1^1 = -0.0894$ and $r_1^2 = 0.0970$. As a result of this loss in security 1 and gain in security 2, the optimal portfolio for period 2 is revised to $w_1^{1*} = 0.002$, $w_1^{2*} = 0.963$ and $w_1^{3*} = 0.035$. Note that since we observe a loss on security 1 and the return on risk-free asset is 0.02, allocation is shifted to security 2 after the first period.

In Fig. 4, we present expected utility surface associated with the case 1 of Table 2. This corresponds to the expectation taken at the random node O_2 of the decision tree and it is obtained as fitting the surface $L_2(W_0^2, W_1^2, r_1)$ to the points (W_0^2, W_1^2, r_1, U_2) in Step 4 of the algorithm. Note that the optimal portfolio $w_1^{1*} = 0.002$ and $w_1^{2*} = 0.963$ is obtained by maximizing this ex-

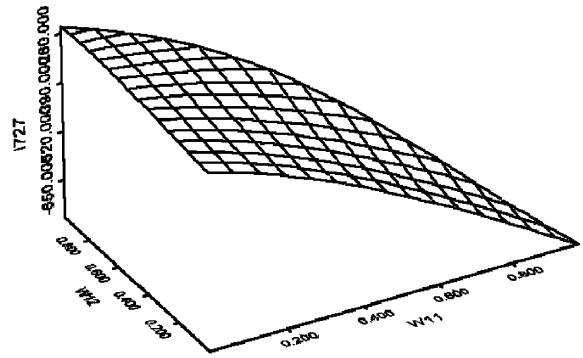


Fig. 4. Expected utility surface for case 1 as a function of w_1^1 and w_1^2 .

pected utility surface with respect to w_1^1 and w_1^2 at the decision node D_1 as implied by the Step 5 of the algorithm.

Similar analysis can be done by using the multivariate ARCH(1) model with a diagonal structure using the priors presented earlier. Note that in this case prior mean return of the first security is higher than the second one. Based on the surface fitting approach with $S = 1000$ samples we have obtained the first period portfolio as $w_0^{1*} = 0.674$, $w_0^{2*} = 0.321$ implying that $w_0^3 = 0.005$. Use of the standard Monte Carlo approach with 1000^4 samples resulted in very similar results again implying the accuracy of the surface fitting algorithm. These results are shown in Table 3.

Two examples of optimal portfolios W_1^{3*} for period two for selected values of W_0^3 and r_1 are shown in Table 4 using the ARCH(1) model. As implied by the comparison with the standard

Table 3
Optimal portfolios using the surface fitting and standard Monte Carlo (SMC) approaches with ARCH(1) model

w_0^{1*}	w_0^{2*}	w_0^{1*} (SMC)	w_0^{2*} (SMC)
0.674	0.321	0.680	0.320

Table 2
Optimal portfolios for period 2 as a function of W_0^3 and r_1 using the surface fitting and standard Monte Carlo (SMC) approaches

	w_0^1	w_0^2	r_1^1	r_1^2	w_1^{1*}	w_1^{2*}	w_1^{1*} (SMC)	w_1^{2*} (SMC)
Case I	0.712	0.007	-0.0894	0.0970	0.002	0.963	0.000	0.960
Case II	0.806	0.165	-0.0552	0.0880	0.011	0.915	0.010	0.910

Table 4
Optimal portfolios for period 2 using the surface fitting and standard Monte Carlo (SMC) approaches ARCH(1) model

	w_0^1	w_0^2	r_1^1	r_1^2	w_1^{1*}	w_1^{2*}	w_1^{1*} (SMC)	w_1^{2*} (SMC)
Case I	0.387	0.578	0.0209	-0.0283	0.853	0.000	0.850	0.000
Case II	0.303	0.576	-0.0205	0.0089	0.002	0.077	0.000	0.080

Monte Carlo algorithm results, the surface fitting algorithm looks very accurate. Note that under the first case in the table, a portfolio of $w_0^1 = 0.387$ and $w_0^2 = 0.578$ is specified implying that $w_0^3 = 0.035$ is allocated to the risk-free asset. As a result of loss in security 2 and gain in security 1 during the first period, the optimal portfolio for period 2 is revised to $w_1^{1*} = 0.853$, $w_1^{2*} = 0.000$ and $w_1^{3*} = 0.147$.

5.1. Sensitivity results and comparison with single-period problem

An important parameter in our implementation of the proposed surface fitting approach is S . Note that specification of S determines the decision space, that is, the number of different feasible portfolios as well as the simulation effort. For computational efficiency, it is desirable to obtain accurate results with a small value of S . We have investigated the robustness of the optimal portfolio using different values of S in the examples that are presented here and have found out that the optimal portfolios are very similar and accurate (when compared to standard Monte Carlo results) for $S = 500, 1000, 2000$.

We have also investigated the sensitivity to prior parameters. It is clear that the optimal allocation for period 1 depends on the choice of the prior parameters in the two multi-variate random variance models. Especially, the allocation is expected to be quite sensitive to changes in parameters associated with mean and precision of security returns. To investigate this we have looked at how the optimal allocation in period 1 changes in the state space model when we change the prior parameters of mean return vector μ_0 and the precision matrix Φ_0 . More specifically we have investigated the sensitivity of the results to changes in m_0 and P_0 .

In the example that we have presented in the above the prior mean vector is specified as

$m_0 = [0.05 \ 0.05]'$ implying equal expected returns for the two risky securities. As expected, keeping all the other prior specifications at the same levels, when we have increased the prior expected return on the first security, optimal allocation to the first security has increased. An example of this is given in Table 5.

Similarly, we can look at changes in the prior scale matrix P_0 for the precision of returns. We note that the properties of the Wishart distribution and (3.8) imply that $E(\Phi_0^{-1} | D_0) = E(\Phi_1^{-1} | D_0) \propto P_0$, that is, the expected conditional variance (or the inverse precision) is proportional to the scale matrix P_0 . Thus, increasing any of the diagonal elements of P_0 implies that the expected variance of the corresponding security return will increase. As expected this implies a shift in allocation to the less riskier security. We have investigated this by increasing the first security's prior parameter in P_0 from 0.02 to 0.05 and have observed that the original allocation of ($w_0^1 = 0.352, w_0^2 = 0.598$) has changed to ($w_0^1 = 0.205, w_0^2 = 0.677$). In other words, the increase in uncertainty about security 1 return has resulted in a decrease in allocation to this asset. Similar insights can be obtained in the multi-variate GARCH model by changing the prior parameters m_0 and C_0 of the model.

In addition to sensitivity analysis, we have also investigated how the optimal portfolios obtained using the two-period model differ from those obtained using the single-period model. In so doing, we have compared the optimal portfolios that are obtained for the first period under the two models using the state space volatility model. Note

Table 5
Sensitivity of optimal portfolios to m_0 in the State Space Model

m_0	w_0^{1*}	w_0^{2*}
$[0.05 \ 0.05]'$	0.352	0.598
$[0.10 \ 0.05]'$	0.551	0.423
$[0.15 \ 0.05]'$	0.629	0.194

Table 6
Comparison of optimal portfolios for the single and two-period problems

Problem	w_0^{1*}	w_0^{2*}
Two-period	0.205	0.677
Single-period	0.322	0.583

that the single period problem does not take into account any uncertainty associated with the second period returns and thus we expect that the optimal portfolios chosen will be different under the two models. The difference between the two sets of optimal portfolios depends on the degree of prior uncertainty as reflected by the parameters. In our example we have assumed the prior mean return vector $\mathbf{m}_0 = [0.05 \ 0.05]'$ and the prior precision scale matrix $\mathbf{P}_0 = \begin{bmatrix} 0.020 & 0.002 \\ 0.002 & 0.010 \end{bmatrix}$ with degrees of freedom $n_0 = 1000$. It can be seen from (3.8) that these prior specifications imply a high degree of expected precision for both security returns for periods 1 and 2. Thus, as a result the optimal portfolios that are obtained under the two models are not expected to differ much. The optimal portfolio under the single period model is obtained as ($w_0^{1*} = 0.398$, $w_0^{2*} = 0.583$) which is slightly different than the optimal portfolio given in Table 1 for the two-period problem.

When we decrease the precision of the first security by specifying the scale matrix as $\mathbf{P}_0 = \begin{bmatrix} 0.050 & 0.002 \\ 0.002 & 0.010 \end{bmatrix}$, we observe a more pronounced difference between the two portfolios as shown in Table 6. Similarly, changing the prior mean vector significantly yields more different optimal portfolios under the two models. For example, if the prior mean is specified as $\mathbf{m}_0 = [0.15 \ 0.05]'$ then the optimal portfolio under the single-period model is obtained as ($w_0^{1*} = 0.533$, $w_0^{2*} = 0.278$) which is quite different than the portfolio ($w_0^{1*} = 0.629$, $w_0^{2*} = 0.194$) obtained under the two-period problem as shown in Table 5.

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