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Abstract

The Ehrenfest urn is a model for the mixing of gases in two chambers. Classic research deals with this system as a Markovian model with a fixed number of balls, and derives the steady-state behavior as a binomial distribution (which can be approximated by a normal distribution). We study the gradual change for an urn containing n balls from the initial condition to the steady state. We look at the status of the urn after k_n draws. We identify three phases of k_n: The growing sublinear, the linear, and the superlinear. In the growing sublinear phase the amount of gas in either chamber is normally distributed, with parameters that are influenced by the initial conditions. In the linear phase a different normal distribution applies, in which the influence of the initial conditions is attenuated. The steady state is not a good approximation until a superlinear amount of time has elapsed. At the superlinear stage the mix is nearly perfect, with a nearly perfect symmetrical normal distribution in which the effect of the initial conditions is completely washed away. We give interpretations for how the results in different phases conjoin at the "seam lines." The Gaussian results are obtained via martingale theory.

1 The Ehrenfest urn as a model for gas mixing

The Ehrenfest urn was first proposed as a model for the mixing of nonreacting gases [4]. We deal here with the speed of this mixing across time phases. The model is for two chambers (say A and B) containing gases (possibly the same). The two chambers are connected through a pipe controlled by a valve. The valve is opened at time 0 and the mixing proceeds over epochs of time, which we can take as the unity. In each time unit (mixing step) one molecule of gas randomly chosen from the population of molecules in both chambers jumps from its chamber to the other one. This continual switching of sides affects a gradual mixing; inducing change in the amount of gas in each chamber. It is of interest to know the amount of gas (number of molecules) in chamber A after a certain period of time.

This physical model of gas mixing can be visualized in terms of a scheme of drawing balls from an urn. We can think of the molecules in chamber A as balls of a certain color (say white) and those in chamber B as balls of an antithetical color (say red). The gas model with n molecules can then be viewed as n balls of two colors all residing in one urn, which evolves in the following manner. At each discrete point in time, we pick a ball at random from the urn. We paint that ball with the opposite color and put it back in the urn. In this equivalent model, the interest is to know the number of white balls (the amount of gas in Chamber A) after a certain period of time.

The classic research deals with this system as a Markovian model with a fixed number of balls, and derives the steady-state behavior as a binomial distribution; see [1], [2] and [6]; for an overview see [8]. In a physical system the number of gas molecules is very large, we shall take it to be n, and is apportioned as $\lfloor \alpha n \rfloor \sim \alpha n$ in chamber A and $n - \lfloor \alpha n \rfloor \sim (1-\alpha)n$ in Chamber B, for some $\alpha \in (0,1)$. One is interested in knowing the behavior of the gases after a certain finite interval of time. So, the question is *How many white balls are in the urn after $k = k_n$ draws, for functions k_n of various growth rates?*

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2 Scope

We shall identify three phases of k_n:

(a) The sublinear phase, when $k_n = o(n)$;

(b) The linear phase, when $k_n \sim \lambda_n n$, for some $\lambda_n > 0$ of a magnitude bounded from above and below;

(c) The superlinear phase, when $n = o(k_n)$.

We shall prove the following general trends. Trivially, at the very low end of the sublinear phase, when $k_n = O(1)$, as $n \to \infty$, there is not much change in the content of the two chambers, only a finite perturbation on the initial conditions can be felt. However, when enough time has elapsed, that is when k_n grows sublinearly to ∞, one sees normal behavior in the amount of gas in each chamber, even for a fairly slowly growing function k_n. We call the phase when k_n grows sublinearly to ∞, the *growing sublinear phase*. Functions that are asymptotically as small as $\frac{1}{20} \ln \ln n$, for example, are sufficient to give a normally distributed mix in each chamber. For the sublinear phase, the initial conditions persist, and the asymptotic normal result in this case contains the initial condition α.

Theorem 1 *Let W_{k_n} be the number of white balls in the Ehrenfest urn (molecules in Chamber A) after k_n draws (gas mixing steps) from an urn with n balls, of which initially the number of white balls is $\lfloor \alpha n \rfloor$, where k_n is in the growing sublinear phase. Then,*

$$\frac{W_{k_n} - n\Big(\frac{1}{2} + \Big(\alpha - \frac{1}{2}\Big)\Big(\frac{n-2}{n}\Big)^{k_n}\Big)}{\sqrt{k_n}} \xrightarrow{\mathcal{D}} \mathcal{N}\big(0, 4\alpha(1-\alpha)\big).$$

Normality continues to hold in the linear and superlinear phases. However, in each phase we get a different normal distribution. The mean and scale factors are essentially different. In the linear phase a different normal distribution (in the usual style of central limit theorems) is in effect, and the parameters of the distribution depend on both the initial condition α and the coefficient of linearity. However, the influence of the initial conditions is attenuated as we get deeper in the linear phase.

Theorem 2 *Let W_{k_n} be the number of white balls in the Ehrenfest urn (molecules in Chamber A) after k_n draws (gas mixing steps) from an urn with n balls, of which initially the number of white balls is $\lfloor \alpha n \rfloor$, when k_n is in the linear phase, where $k_n \sim \lambda_n n$, for some λ_n such that $0 < S_1 \le \lambda_n \le S_2 < \infty$. Then,*

$$\frac{W_{k_n} - \Big(\Big(\alpha - \frac{1}{2}\Big)e^{-2\lambda_n} + \frac{1}{2}\Big)\, n}{\sqrt{\frac{e^{4\lambda_n} - 1 - 4\lambda_n(2\alpha-1)^2}{4e^{4\lambda_n}}\, n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

As one might expect, after a very long period of time, as in the superlinear case, the mixing is nearly complete, and the result is a central limit theorem in which the effect of any initial conditions is washed away.

Theorem 3 *Let W_{k_n} be the number of white balls in the Ehrenfest urn (molecules in Chamber A) after k_n draws (gas mixing steps) from an urn with n balls, of which initially the number of white balls is $\lfloor \alpha n \rfloor$, where k_n is in the superlinear phase. Then,*

$$\frac{W_{k_n} - \frac{1}{2}n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{1}{4}\Big).$$

To put our results in perspective note that these Gaussian laws in different phases consider the diffusion of gas when the chambers contain a large number of particles, tending to infinity. Some earlier research considers the transience in Ehrenfest models for a fixed number of particles, where the case is a finite Markov chain possessing a stationary distribution. Discussion of the variation distance from the stationary distribution is given in [3], where the diffusion of a fixed number of particles is considered over an extended number of draws. A cutoff phenomenon is reported in [3]: The total variation distance stays high (close to 1) up until a logarithmic number of draws when it drops sharply to low values near 0.

3 Organization

The rest of this paper has the following organization. In Section 4 we set up exact formulas, starting from an exact stochastic recurrence and going through an exact calculation of the mean and variance of the number of white balls after n sample draws. In Section 5 we derive the underlying martingale. In Section 6 we discuss the three phases: the growing sublinear, the linear, and the superlinear, where a subsection is devoted for each phase. In these subsections we prove the announced results.

Throughout, we shall use the following standard probability notation. We shall denote the normally distributed random variate with mean 0 and variance ν^2 by $\mathcal{N}(0, \nu^2)$. We shall use the symbols $\xrightarrow{\mathcal{D}}$ and $\xrightarrow{\mathcal{P}}$ respectively for convergence in distribution and in probability. The notation $O_{\mathcal{L}_1}(g(n))$ will stand for a sequence of random variables that is $O(g(n))$ in the $\mathcal{L}_1$ norm, that is, when we describe a sequence of random variables X_n to be $O_{\mathcal{L}_1}(g(n))$, we mean there exist a positive constant C and a positive integer n_0, such that $\mathbf{E}[|X_n|] \le C|g(n)|$, for all $n \ge n_0$. We let $\mathcal{F}_j$ be the sigma field generated by the first j draws.

Unless otherwise stated, all asymptotics will mean asymptotic equivalents and bounds as $n \to \infty$. The number $n/(n-2)$ will appear often, and we shall give it the designation ρ_n. We shall repeatedly use well-known facts about ρ_n^{yn}, for $y > 0$, such as the fact that ρ_n^{yn} is asymptotically $e^{2y} + O(1/n)$.

We shall also need the backward difference operator ∇, which when applied to a function $h(i)$, with integer argument i, gives the difference between two successive steps; that is, $\nabla h(i) = h(i) - h(i-1)$.

4 Exact moments

Let $W_j = W_j(n)$ be the number of white balls (molecules in Chamber A) after j such draws (mixing steps). Let I_n^W and I_n^R respectively be the indicators of picking a white or a red ball in the nth step. Because of their mutual exclusion, we have $I_n^R = 1 - I_n^W$. There is stochastic dependence between W_{j-1} and W_j. After $j-1$ draws, the number of white balls in the urn is W_{j-1}, and the number of white balls will increase by 1 after one drawing, if a red ball is picked, but will decrease by 1, if a white ball is picked. And so,

$$W_j = W_{j-1} + I_n^R - I_n^W = W_{j-1} + 1 - 2I_n^W. \tag{1}$$

It follows from the stochastic recurrence (1) that

$$\begin{aligned} \mathbf{E}[W_j] &= \mathbf{E}[W_{j-1}] + 1 - 2\mathbf{E}\big[\mathbf{E}[I_n^W \mid \mathcal{F}_{j-1}]\big] \\ &= \mathbf{E}[W_{j-1}] + 1 - 2\mathbf{E}\Big[\frac{W_{j-1}}{n}\Big] \\ &= \Big(1 - \frac{2}{n}\Big)\mathbf{E}[W_{j-1}] + 1. \end{aligned}$$

This recurrence can be solved by unwinding it all the way back to the initial condition W_0. One gets

$$\mathbf{E}[W_j] = \frac{1}{2}n + \Big(W_0 - \frac{n}{2}\Big)\Big(\frac{n-2}{n}\Big)^j. \tag{2}$$

Note that under the assumption that $W_0 = \lfloor \alpha n \rfloor \sim \alpha n$, the mean after k_n mixing steps experiences phases according to how fast k_n grows. For the growing sublinear, linear and superlinear phases we have the mean asymptotics

$$\mathbf{E}[W_{k_n}] \sim \begin{cases} \alpha\, n, & \text{for } k_n = o(n); \\ \Big(\Big(\alpha - \frac{1}{2}\Big)e^{-2\lambda_n} + \frac{1}{2}\Big)\, n, & \text{for } k_n \sim \lambda_n n; \\ \frac{1}{2}\, n, & \text{for } n = o(k_n). \end{cases}$$

Observe how the three phases meet on average at the seam lines. The linear phase with $\lambda_n = 0$ gives the result in the growing sublinear phase, and with $\lambda_n = \infty$ gives the result of the superlinear pase.

Toward a computation of the second moment we write (1) in its squared form

$$W_j^2 = (W_{j-1} + 1 - 2I_n^W)^2 = W_{j-1}^2 + 2W_{j-1}(1 - 2I_n^W) + 1.$$

We follow similar steps as those for the mean: take expectations to obtain a recurrence for the second moment and solve it. The variance follows

$$\begin{aligned} \mathbf{Var}\big[W_j\big] &= \mathbf{E}[W_j^2] - \big(\mathbf{E}[W_j]\big)^2 \\ &= \frac{1}{4}n + \Big(\Big(\frac{n}{2} - W_0\Big)^2 - \frac{n}{4}\Big)\Big(\frac{n-4}{n}\Big)^j \\ &\qquad - \Big(\frac{n}{2} - W_0\Big)^2\Big(\frac{n-2}{n}\Big)^{2j}. \end{aligned} \tag{3}$$

Like the mean, under the assumption that $W_0 = \lfloor \alpha n \rfloor \sim \alpha n$, the variance of the amount of gas in Chamber A after k_n mixing steps experiences phases according to how fast k_n grows. For the growing sublinear, linear and superlinear phases we have the variance asymptotics

$$\mathbf{Var}[W_{k_n}] \sim \begin{cases} 4\alpha(1-\alpha)\,k_n, & \text{for } k_n = o(n); \\ \frac{e^{4\lambda_n}-1-4\lambda_n(2\alpha-1)^2}{4e^{4\lambda_n}}\,n, & \text{for } k_n \sim \lambda_n n; \\ \frac{1}{4}\,n, & \text{for } n = o(k_n). \end{cases}$$

5 A martingale underlying gas mixing

Conditioning the recurrence (1) on the content of the sigma field $\mathcal{F}_{j-1}$, one gets

$$\mathbf{E}[W_j \,|\, \mathcal{F}_{j-1}] = \Big(1 - \frac{2}{n}\Big) W_{j-1} + 1.$$

There is an associated martingale as in the following lemma.

Lemma 1

$$M_n := \rho_n^j W_j - \frac{\rho_n^{j+1} - \rho_n}{\rho_n - 1}$$

is a martingale, where $\rho_n = n/(n-2)$.

Proof.
Introduce the transformation

$$M_j = a_j W_j + b_j.$$

We wish to turn M_n into a martingale with suitable choices of deterministic sequences a_n and b_n. So, M_n must satisfy

$$\text{(4)} \quad \mathbf{E}[M_j \,|\, \mathcal{F}_{j-1}] = M_{j-1} = a_{j-1} W_{j-1} + b_{j-1}.$$

We compute

$$\begin{aligned} \mathbf{E}[M_j \,|\, \mathcal{F}_{j-1}] &= \mathbf{E}[a_j W_j + b_j \,|\, \mathcal{F}_{j-1}] \\ &= a_j \mathbf{E}[W_j \,|\, \mathcal{F}_{j-1}] + b_j. \end{aligned}$$

From (1) we proceed with

$$\mathbf{E}[M_j \,|\, \mathcal{F}_{j-1}] = a_j \Big(1 - \frac{2}{n}\Big) W_{j-1} + a_j + b_j.$$

Matching the coefficients of this equality with those in (4), we arrive at recurrences for a_j and b_j. We have $a_j = \rho_n a_{j-1}$. This recurrence unfolds easily to give $a_j = \rho_n^j a_0$, for any arbitrary value of a_0; we take $a_0 = 1$.

We also have the recurrence $b_j = b_{j-1} - a_j$, which unwinds into

$$b_j = b_0 - \sum_{k=1}^{j} \rho_n^k,$$

for arbitrary b_0; we take $b_0 = 0$ and simplify the sum to

$$b_j = -\frac{\rho_n^{j+1} - \rho_n}{\rho_n - 1}. \quad \blacksquare$$

The fact that M_j is a martingale is key to proving Gaussian limits in all the phases. We shall deal with the centered martingale

$$\tilde{M}_j = M_j - W_0$$

(which has mean 0) to employ the martingale central limit theorem, which requires calculations on a zero-mean martingale. Sufficient conditions for the central limit theorem for a zero-mean martingale $X_{j,n}$ are the conditional Lindeberg's condition and the conditional variance condition on the martingale differences $\nabla X_{j,k_n} = X_{j,k_n} - X_{j-1,k_n}$; (see Theorem 3.2 and Corollary 3.1, P. 58 in [5]).

Specifically in our case, the conditional Lindeberg's condition requires that, for some positive increasing sequence ξ_n, and for all $\varepsilon > 0$,

$$\begin{aligned} U_n \quad &:= \quad \sum_{j=1}^{k_n} \mathbf{E}\Big[\Big(\frac{\nabla \tilde{M}_j}{\xi_n}\Big)^2 \mathbf{1}_{\big\{\big|\frac{\nabla \tilde{M}_j}{\xi_n}\big| > \varepsilon\big\}} \,\Big|\, \mathcal{F}_{j-1}\Big] \\ \text{(5)} \quad &\xrightarrow{\mathcal{P}} \quad 0, \end{aligned}$$

where the indicator $\mathbf{1}_{\mathcal{E}}$ is a function of a sample space that assumes the value 1 if $\mathcal{E}$ occurs, and otherwise it assumes the value 0, and, for a constant c, a c–conditional variance condition requires that

$$(6) \qquad V_n := \sum_{j=1}^{k_n} \mathbf{E}\Big[\Big(\frac{\nabla \tilde M_j}{\xi_n}\Big)^2 \,\Big|\, \mathcal{F}_{j-1}\Big] \xrightarrow{\mathcal{P}} c.$$

When both conditions hold, the sum $\sum_{j=1}^{k_n} \nabla \tilde M_j/\xi_n = (M_{k_n} - M_0)/\xi_n = (M_{k_n} - W_0)/\xi_n$ converges to the normally distributed random variable $\mathcal{N}(0, c^2)$.

To derive a martingale central limit theorem in any of the phases, we need to identify the appropriate scale ξ_n for that phase. For calculations involved in conditional Lindeberg's condition the following lemma is helpful in all the phases.

Lemma 2

$$|\nabla \tilde M_j| \le 4\rho_n^j.$$

Proof.

With the help of (1) write the absolute differences as

$$\begin{aligned}
|\nabla \tilde M_j| &= \big|(M_j - W_0) - (M_{j-1} - W_0)\big| \\
&= \big|(\rho_n^j W_j + b_j) - (\rho_n^{j-1} W_{j-1} + b_{j-1})\big| \\
&= \big|\big(\rho_n^j (W_{j-1} + I_j^R - I_j^W) + b_j\big) \\
&\qquad -(\rho_n^{j-1} W_{j-1} + b_{j-1})\big| \\
&\le \rho_n^{j-1}\big((\rho_n - 1)W_{j-1} + \rho_n |I_j^R - I_j^W| + \rho_n\big) \\
&\le \rho_n^{j-1}\Big(\frac{2}{n-2}\, W_{j-1} + \frac{2n}{n-2}\Big).
\end{aligned}$$

The number of white balls at any stage is at most n, and the lemma follows. ■

For calculations involved in conditional Lindeberg's condition we need $\mathbf{E}\Big[(\nabla \tilde M_j)^2 \,|\, \mathcal{F}_{j-1}]$. After some laborious calculation involving (1), we get

$$(7)\ V_n = -\frac{4}{n^2\xi_n^2} \sum_{j=1}^{k_n} \rho_n^{2j} W_{j-1}^2 + \frac{4}{n\xi_n^2} \sum_{j=1}^{k_n} \rho_n^{2j} W_{j-1}.$$

6 Phases during long-term drawing

Suppose the gas mixing process is perpetuated indefinitely. We shall see that as the ball drawing continues from the Ehrenfest urn the process experiences different phases.

6.1 The growing sublinear phase

Let k_n be in the growing sublinear phase (k_n grows to ∞, and $k_n = o(n)$). The number of white balls after $0 \le j \le k_n$ draws has obvious bounds—if all the draws are of red balls an increase by j goes in favor of the number of white balls over their initial number, and if all the draws are of white balls a deficit of j occurs against the initial number of white balls. We have the inequalities

$$W_0 - j \le W_j \le W_0 + j.$$

We can ascertain that

$$(8) \qquad W_j = \alpha n + O(k_n),$$

for all $0 \le j \le k_n$.

Consider $1 \le j \le k_n = o(n)$. For n large enough (greater than some $N_0 > 2$),

$$\rho_n^j = \Big(\frac{n}{n-2}\Big)^j \le 2.$$

It follows from Lemma 2 that

$$|\nabla \tilde M_j| \le 8.$$

Proof of Theorem 1.

Recall the expressions for U_n (cf. (5)), and V_n (cf. (6)). For the growing sublinear phase, take the scale factor $\xi_n = \sqrt{k_n}$. The proof will be complete if we show that U_n converges to 0 in probability, and V_n converges to a constant in probability.

For the conditional Lindeberg's condition we have the uniform upper bound of 8 for $|\nabla \tilde M_j|$, for all n greater than some $N_0 > 2$ (as discussed right before this proof). Therefore, for any $\varepsilon > 0$, in the expression

$$U_n = \sum_{j=1}^{k_n} \mathbf{E}\Big[\Big(\frac{\nabla \tilde M_j}{\sqrt{k_n}}\Big)^2 \mathbf{1}_{\Big\{\Big|\frac{\nabla \tilde M_j}{\sqrt{k_n}}\Big|>\varepsilon\Big\}}\,\Big|\, \mathcal{F}_{j-1}\Big]$$

the sets $\{|\nabla \tilde M_j| > \varepsilon\sqrt{k_n}\,\}$ are all empty, for all n greater than some $n_0(\varepsilon) > N_0$. For large n we have

$$\begin{aligned} U_n &= \sum_{j=1}^{n_0(\varepsilon)} \mathbf{E}\Big[\Big(\frac{\nabla \tilde M_j}{\sqrt{k_n}}\Big)^2 \mathbf{1}_{\Big\{\Big|\frac{\nabla \tilde M_j}{\sqrt{k_n}}\Big|>\varepsilon\Big\}}\,\Big|\, \mathcal{F}_{j-1}\Big] \\ &\le \frac{1}{k_n}\sum_{j=1}^{n_0(\varepsilon)} \mathbf{E}\big[(\nabla \tilde M_j)^2 \,|\, \mathcal{F}_{j-1}\big] \\ &\le \frac{64 n_0(\varepsilon)}{k_n} \\ &\to 0, \qquad \text{as } n \to \infty. \end{aligned}$$

The conditional Lindeberg's condition has been verified throughout the growing sublinear phase.

In (7) replace W_{j-1} by the asymptotic equivalent in (8) to get

$$\begin{aligned} V_n &= -\frac{4}{n^2\xi_n}\sum_{j=1}^{k_n} \rho_n^{2j}\big(\alpha n + O(k_n)\big)^2 \\ &\qquad + \frac{4}{n\xi_n}\sum_{j=1}^{k_n} \rho_n^{2j}\big(\alpha n + O(k_n)\big) \\ &= \frac{1}{k_n}\Big(4\alpha(1-\alpha) + O\Big(\frac{k_n}{n}\Big) + O\Big(\frac{k_n^2}{n^2}\Big)\Big)\sum_{j=1}^{k_n} \rho_n^{2j}. \end{aligned}$$

Recall that $\rho_n = n/(n-2)$, and we can simplify the remaining sum asymptotically as

$$\begin{aligned} \sum_{j=1}^{k_n} \rho_n^{2j} &= \frac{\Big(\frac{n}{n-2}\Big)^{2k_n+2} - 1}{\Big(\frac{n}{n-2}\Big)^2 - 1} - 1 \\ &= \frac{(n-2)^2}{4n-4}\Big(e^{(2k_n+2)\ln\left(\frac{n}{n-2}\right)} - 1\Big) - 1 \\ &= \frac{(n-2)^2}{4n-4}\Big(\Big(1 + \frac{4k_n}{n} + O\Big(\frac{k_n^2}{n^2}\Big)\Big) - 1\Big) - 1 \\ &= k_n + O\Big(\frac{k_n^2}{n}\Big). \end{aligned}$$

And so, we have

$$\begin{aligned} V_n &= \frac{1}{k_n}\Big(4\alpha(1-\alpha) + O\Big(\frac{k_n}{n}\Big) + O\Big(\frac{k_n^2}{n^2}\Big)\Big) \\ &\qquad \times \Big(k_n + O\Big(\frac{k_n^2}{n}\Big)\Big) \\ &= 4\alpha(1-\alpha) + O\Big(\frac{k_n}{n}\Big) + O\Big(\frac{k_n^2}{n^2}\Big) + O\Big(\frac{k_n^3}{n^3}\Big) \\ &\to 4\alpha(1-\alpha). \end{aligned}$$

The $4\alpha(1-\alpha)$–conditional variance condition has been verified in the growing sublinear phase.

With both conditions checked, the martingale central limit theorem gives

$$\sum_{j=1}^{k_n}\Big(\frac{\nabla \tilde M_j}{\sqrt{k_n}}\Big) = \frac{M_{k_n} - W_0}{\sqrt{k_n}} \xrightarrow{\mathcal{D}} \mathcal{N}\big(0, 4\alpha(1-\alpha)\big).$$

Subsequently, we write

$$\frac{\rho_n^{k_n} W_{k_n} - \frac{\rho_n^{k_n+1}-\rho_n}{\rho_n - 1} - W_0}{\sqrt{k_n}} \xrightarrow{\mathcal{D}} \mathcal{N}\big(0, 4\alpha(1-\alpha)\big).$$

With $(n/(n-2))^{-k_n}$ converging to 1 in the sublinear phase and an application of Slutsky's theorem (See [7], 1993, P. 146–147), we arrive at

$$\frac{W_{k_n} - n\Big(\frac{1}{2} + \Big(\alpha - \frac{1}{2}\Big)\Big(\frac{n-2}{n}\Big)^{k_n}\Big)}{\sqrt{k_n}} \xrightarrow{\mathcal{D}} \mathcal{N}\big(0, 4\alpha(1-\alpha)\big). \quad \blacksquare$$

6.2 The linear phase

In the linear phase $k_n \sim \lambda_n n$, for some $\lambda_n > 0$ of a magnitude uniformly bounded from above and below, that is, for two positive constants, S_1 and S_2, and for all n,

$$S_1 \le \lambda_n \le S_2.$$

At this phase of the gas mixing we have the following asymptotic equivalents (as $n \to \infty$), following from (2) and (3):

$$\mathbf{E}[W_{k_n}] = \mu_n n + O(1), \tag{9}$$

and

(10) $$\mathbf{Var}[W_{k_n}] \sim v_n n + O(1),$$

where

$$\mu_n = \Big(\alpha - \frac{1}{2}\Big)e^{-2\lambda_n} + \frac{1}{2},$$

and

$$v_n = \frac{e^{4\lambda_n} - 1 - 4\lambda_n(1-2\alpha)^2}{4e^{4\lambda_n}} = O(1).$$

We start with a first-order result for W_{k_n}.

Theorem 4 *For $k_n = \lambda_n n + o(n)$, for some $\lambda_n > 0$ of a magnitude bounded from above and below,*

$$\frac{W_{k_n}}{\Big(\Big(\alpha - \frac{1}{2}\Big)e^{-2\lambda_n} + \frac{1}{2}\Big)\,n} \xrightarrow{\mathcal{P}} 1.$$

Proof.

By Chebyshev's inequality,

$$\begin{aligned}\mathbf{Prob}\big(\big|W_{k_n} - \mathbf{E}[W_{k_n}]\big| \ge \varepsilon\mathbf{E}[W_{k_n}]\big) &\le \frac{\mathbf{Var}[W_{k_n}]}{\varepsilon^2(\mathbf{E}[W_{k_n}])^2}\\ &\sim \frac{v_n n}{\varepsilon^2\mu_n^2 n^2}\\ &\to 0, \qquad \text{as} \quad n\to\infty.\end{aligned}$$

Hence,

$$\frac{W_{k_n}}{\mathbf{E}[W_{k_n}]} \xrightarrow{\mathcal{P}} 1.$$

From the convergence $\mathbf{E}[W_{k_n}]/(\mu_n n) \to 1$, and Slutsky's Theorem in its multiplicative form (cf. [7], 1993, P. 147), we obtain

$$\frac{W_{k_n}}{\mu_n n} \xrightarrow{\mathcal{P}} 1. \quad \blacksquare$$

Before we dwell on the proof of a central limit theorem for the amount of gas in Chamber A by the end of some linear phase, we need a technical lemma, which shows that W_{k_n} grows linearly with n, like its mean, with correction terms that are $O_{\mathcal{L}_1}(\sqrt{n}\,)$. The purpose of this calculation is for later summation to verify conditional Lindeberg's condition.

Lemma 3 *Let W_{k_n} be the number of white balls in the urn after k_n draws, where $k_n = \lambda_n n + o(n)$, for some λ_n, such that $0 < S_1 \le \lambda_n \le S_2 < \infty$. Then*

$$W_{k_n} = \mu_n n + O_{\mathcal{L}_1}(\sqrt{n}\,),$$

and

$$W_{k_n}^2 = \mu_n^2 n^2 + O_{\mathcal{L}_1}(n^{3/2}),$$

Proof.

From the asymptotics of the mean and variance, as given in (9) and (10), for large n we have

$$\begin{aligned}\mathbf{E}\big[\big(W_{k_n} - \mu_n n\big)^2\big] &= \mathbf{Var}[W_{k_n}] + \big(\mathbf{E}[W_{k_n}] - \mu_n n\big)^2\\ &= v_n\, n + O(1)\\ &\le \frac{e^{4S_2} - 1 - 4S_1(1-2\alpha)^2}{4e^{4S_1}}\, n\\ &\qquad +O(1)\\ &= O(n). \qquad (11)\end{aligned}$$

So, by Jensen's inequality

$$\mathbf{E}\Big[\Big|W_{k_n} - \mu_n n\Big|\Big] \le \sqrt{\mathbf{E}\big[\big(W_{k_n} - \mu_n n\big)^2\big]} = O(\sqrt{n}\,),$$

which implies

$$W_{k_n} = \mu_n n + O_{\mathcal{L}_1}(\sqrt{n}\,).$$

Moreover, by the Cauchy-Schwarz inequality we have

$$\begin{aligned}\mathbf{E}\Big[\Big|W_{k_n}^2 - \mu_n^2 n^2\Big|\Big] &= \mathbf{E}\Big[\Big|W_{k_n} + \mu_n n\Big|\Big|W_{k_n} - \mu_n n\Big|\Big]\\ &\le \sqrt{\mathbf{E}\big[\big(W_{k_n} + \mu_n n\big)^2\big]\,\mathbf{E}\big[\big(W_{k_n} - \mu_n n\big)^2\big]}\,.\end{aligned}$$

Obviously, $W_{k_n} + \mu_n n \le n + ((\alpha - \frac{1}{2})e^{-2S_1} + \frac{1}{2})\,n + O(1) = O(n)$. We employ (11) to bound $\sqrt{\mathbf{E}\big[\big(W_{k_n} - \mu_n n\big)^2\big]}$ by $O(\sqrt{n}\,)$. Subsequently, we obtain

$$\mathbf{E}\Big[\Big|W_{k_n}^2 - \mu_n^2 n^2\Big|\Big] = O\big(n^{3/2}\big),$$

which implies

$$W_{k_n}^2 = \mu_n^2 n^2 + O_{\mathcal{L}_1}\big(n^{3/2}\big). \quad \blacksquare$$

Proof of Theorem 2.

Recall the expressions for U_n (cf. (5)), and V_n (cf. (6)). In this phase we take the scale factor ξ_n to be $\sqrt{4nv_n e^{4\lambda_n}}$. The proof will be complete if we show that U_n converges to 0 in probability, and V_n converges to a constant in probability.

Suppose $j \sim yn$, with $0 < y < S_2$. It follows from Lemma 2 that

$$|\nabla \tilde{M}_j| \le 8e^{2S_2},$$

for all n greater than sum $N_0' > 2$. In view of this uniform bound, the conditional Lindeberg's condition can be argued as follows. The set $\{|\nabla \tilde{M}_j| > \varepsilon\sqrt{4nv_n e^{4\lambda_n}}\}$ is empty, for all n greater than some $n_0(\varepsilon) > N_0' > 2$. For large n, we have

$$\begin{aligned} U_n &= \sum_{j=1}^{n_0(\varepsilon)} \mathbf{E}\Big[\Big(\frac{\nabla \tilde{M}_j}{\sqrt{4nv_n e^{4\lambda_n}}}\Big)^2 \\ &\qquad\qquad \times \mathbf{1}_{\Big\{\Big|\frac{\nabla \tilde{M}_j}{\sqrt{4nv_n e^{4\lambda_n}}}\Big|>\varepsilon\Big\}} \,\Big|\, \mathcal{F}_{j-1}\Big] \\ &\le \frac{1}{4nv_n e^{4\lambda_n}} \sum_{j=1}^{n_0(\varepsilon)} \mathbf{E}\big[(\nabla \tilde{M}_j)^2 \,|\, \mathcal{F}_{j-1}\big] \\ &\le \frac{16e^{2S_2} n_0(\varepsilon)}{nv_n e^{4\lambda_n}} \\ &\to 0, \qquad as\ n \to \infty. \end{aligned}$$

The conditional Lindeberg's condition has been verified in the linear phase.

The asymptotic equivalents in Lemma 3 apply only in the linear phase. However, before the linear phase the obvious bound n on W_{j-1} is sufficient for our purpose.

To asymptotically handle the sums in the conditional Lindeberg's condition (going over the range of indexes 1 to $k_n \sim \lambda_n n$) let us break them up at some point near the beginning of the linear phase. Choose a small positive $\epsilon < S_1$ and break up the sums in V_n into sums going from 1 to $\lfloor \varepsilon n \rfloor - 1$ and sums starting at $\lfloor \varepsilon n \rfloor$ and ending at k_n. Applying the asymptotics of Lemma 3 the $\frac{1}{4}$–conditional variance condition can be verified in the linear phase.

According to the martingale central limit theorem

$$\sum_{j=1}^{k_n} \Big(\frac{\nabla \tilde{M}_j}{\sqrt{4nv_n e^{4\lambda_n}}}\Big) = \frac{M_{k_n} - M_0}{\sqrt{4nv_n e^{4\lambda_n}}} \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{1}{4}\Big).$$

Subsequently, we write

$$\frac{\rho_n^{k_n} W_{k_n} - \frac{\rho_n^{k_n+1} - \rho_n}{\rho_n - 1} - W_0}{\sqrt{nv_n e^{4\lambda_n}}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1).$$

Using the asymptotic relation $\rho_n^{k_n} = (n/(n-2))^{k_n} = e^{2\lambda_n} + O(1/n)$ in the linear phase, and that the initial condition is $W_0 = \lfloor \alpha n \rfloor \sim \alpha n$, the statement of the theorem follows from an application of Slutsky's theorem (See [7], 1993, P. 146–147). ■

6.3 The superlinear phase

Suppose that the gas mixing continued for a long period of time. As seen from the behavior of the average, the initial conditions are attenuated through the linear phase and the fixed average component $\frac{1}{2}n$ becomes more pronounced and eventually dominates in the superlinear phase. Many of the principles of the proof for the linear phase apply within the superlinear phase, so we shall be a bit brief in presenting an adjustment of these proofs. For instance, via the asymptotic equivalents of the mean and variance in the superlinear phase, we can mimic the proof of Theorem 4, and get a similar result. Namely, when $n = o(k_n)$, we have

$$\frac{W_{k_n}}{n} \xrightarrow{\mathcal{P}} \frac{1}{2}.$$

Also, in view of the mean and variance asymptotics we can replicate the result of Lemma 3. We only need to replace μ_n by $\frac{1}{2}$, and the proof goes through verbatim to obtain

$$W_{k_n} = \frac{1}{2}n + O_{\mathcal{L}_1}(\sqrt{n}\,),$$

and

$$W_{k_n}^2 = \frac{1}{4}n^2 + O_{\mathcal{L}_1}(n^{3/2}).$$

In the superlinear phase the sequence ρ_n^j grows very fast, we must take it as part of the normalization. We take the scale ξ_n to be $\rho_n^{k_n}\sqrt{n}$. By Lemma 2 we have

$$\Big|\frac{\nabla \tilde{M}_j}{\rho_n^j}\Big| \le 4.$$

Proof of Theorem 3.

For the conditional Lindeberg's condition we have the uniform upper bound of 4 for $|\nabla \tilde{M}_j \rho_n^{-j}|$, for all n greater than some $N_0 > 2$ (see Lemma 2). Therefore, for any $\varepsilon > 0$, the sets $\{|\nabla \tilde{M}_j \rho_n^{-j}| > \varepsilon\sqrt{n}\}$ are all empty, for all n greater than some $n_0(\varepsilon) > N_0$. For large n we have

$$\begin{aligned} U_n &= \sum_{j=1}^{n_0(\varepsilon)} \mathbf{E}\Big[\Big(\frac{\nabla \tilde{M}_j}{\rho_n^j \sqrt{n}}\Big)^2 \mathbf{1}_{\Big\{\Big|\frac{\nabla \tilde{M}_j}{\rho_n^j \sqrt{n}}\Big| > \varepsilon\Big\}} \Big| \mathcal{F}_{j-1}\Big] \\ &\le \frac{1}{n} \sum_{j=1}^{n_0(\varepsilon)} \mathbf{E}\Big[\Big(\frac{\nabla \tilde{M}_j}{\rho_n^j}\Big)^2 \,|\, \mathcal{F}_{j-1}\Big] \\ &\le \frac{16 n_0(\varepsilon)}{n} \\ &\to 0, \qquad \text{as } n \to \infty. \end{aligned}$$

The conditional Lindeberg's condition has been verified throughout the superlinear phase.

For the sum in the conditional variance condition we apply the bound $W_{j-1} \le n$ until the superlinear phase. More precisely, to asymptotically handle the sums in the conditional Lindeberg's condition (going over the range of indexes 1 to k_n) we break up the sums in V_n into sums going from 1 to $k_n' - 1$, which is any superlinear function of order less than k_n (giving ignorable contribution) and sums starting at k_n' and ending at k_n (most of the contribution comes near k_n). We can take $k_n' = \lfloor k_n / \ln(k_n/n) \rfloor$. The $\frac{1}{4}$–variance condition follows.

According to the martingale central limit theorem

$$\frac{\rho_n^{k_n} W_{k_n} - \frac{\rho_n^{k_n+1} - \rho_n}{\rho_n - 1} - W_0}{\rho_{k_n}^n \sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}\Big(0, \frac{1}{4}\Big).$$

Using that $\rho_n^{k_n} = (n/(n-2))^{k_n}$ grows exponentially in the superlinear phase, an application of Slutsky's theorem (See Karr, 1993, P. 146–147) gives the result. ■

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