

I^2SDS
The Institute for Integrating Statistics in Decision Sciences

Technical Report TR-2007-10
May 29, 2007

Bayesian Models for Accelerated Life Tests

Refik Soyer
Department of Decision Sciences
The George Washington University

Alaattin Erkanli
Developmental Epidemiology Program
Duke University Medical Center

Jason R. Merrick
Department of Statistical Sciences and Operations Research
Virginia Commonwealth University

Article ID EQR 517

BAYESIAN MODELS FOR ACCELERATED LIFE TESTS¹

Refik Soyer

Department of Decision Sciences
School of Business
George Washington University
Washington, DC 20052

Tel: +1 (202) 994 6445
Fax: +1 (202) 994 2736
E-mail: soyer@gwu.edu

Alaattin Erkanli

Developmental Epidemiology Program
Duke University Medical Center
Durham, NC 27710
E-mail: al@psych.mc.duke.edu

Jason R. Merrick

Department of Statistical Sciences and Operations Research
Virginia Commonwealth University
Richmand, VA 23284
E-mail: jrmerric@mail1.vcu.edu

Abstract

In this article we consider accelerated life test models where the effect of accelerated environments on failure behavior is described by time transformation functions. We present extensions of basic models by introducing hierarchical and dynamic strategies to allow for changes in the time transformation function with the accelerated environment. We develop Markov chain Monte Carlo methods to make Bayesian inferences from these models where no analytical forms are available for the posterior and predictive distributions of interest. We illustrate our approach with two examples and discuss what type of insights can be obtained from Bayesian analysis.

Key Words: Posterior and predictive analysis, life testing, hierarchical Bayes, Markov chain Monte Carlo, Gibbs sampler.

¹Contribution to *The Encyclopedia of Statistics in Quality and Reliability* (Eds. F. Ruggeri, R. Kenett and F. Faltin), John Wiley & Sons, UK.

1. Introduction and Overview

In dealing with high reliability components\systems, it is not unusual to conduct tests in a more severe environment than the actual use environment. Such tests are called accelerated life tests (ALT's) and are usually conducted with the purpose of inducing early failures to reduce time and cost of testing. The important statistical issue in ALT's is to make inferences about the failure behavior of the items at the user environment based on failure data obtained under the more severe environments. In recent years there has been an increasing interest in developing Bayesian methods for making inferences from accelerated life tests; see for example Mazzuchi and Soyer (1992), van Dorp et al. (1996), Mazzuchi, Soyer and Vopatek (1997), and van Dorp and Mazzuchi (2004a, 2004b). As a result of using complicated time transformation functions and/or ramping functions in step-stress ALT's, Bayesian inferences from these models have become extremely difficult. To deal with intractable predictive and posterior distributions, the previous approaches used large sample approximations or approximate methods of inference such as the *linear Bayes*. Only a few of the earlier Bayesian work in ALT's, such as Blackwell and Singpurwalla (1988) and Mazzuchi and Soyer (1992), have considered models that allow time transformation functions that change with stress environment.

In this paper we consider Bayesian models that allow dynamic time transformation functions and propose different modeling strategies. We focus specifically on hierarchical Bayes models, similar to those described by Lindley and Smith (1972), and dynamic models, as in Mazzuchi and Soyer (1992), and develop Bayesian computational methods for making inferences using Markov chain Monte Carlo (MCMC) approaches. The attractive feature of the MCMC methods is that they enable generation of samples from the posterior and predictive distributions without having to obtain the exact distributional forms. Thus, unlike the previous approaches, we can perform fully Bayesian analyses by computing all the posterior and predictive distributions of interest.

In Section 2, we present a basic parametric accelerated life testing model assuming a Weibull life distribution and several time-transformation functions. We discuss the difficulties involved in computing the posterior and predictive distributions using conventional techniques and present alternative inference based on MCMC methods. In Section 3, we consider hierarchical and Markov models as novel extensions of the basic parametric model to allow for dynamic time transformation functions and discuss inference and reliability predictions using MCMC methods. In Section 4, we illustrate our approaches using two real life-testing examples, discuss model comparison, and investigate the appropriateness of dynamic time transformation functions.

2. A Parametric ALT Model

An accelerated test environment is created typically by increasing the level of one or more stress variables (temperature, voltage, etc.) to values which are higher than those at normal operating conditions. For simplicity, in our development we assume that the environment is defined by a single stress variable. Extension to the multivariable case is straightforward. To introduce some notation, let S_i denote the level of the stress variable at the i th accelerated test environment and assume that test will be conducted at K accelerated levels of the stress variable which are specified in advance. As noted before, our main objective is to make predictive inferences about the failure behavior of items at the use stress environment, S_u , based on the data from the K accelerated testing environments where $S_i > S_u$ for $i = 1, 2, \dots, K$. It is also possible to incorporate testing at the use environment with the proposed procedure, but in our development we assume that no testing is conducted at S_u .

We assume that under the i th accelerated test environment, the failure behavior of the items can be described by a Weibull model with density

$$p(x_i | \lambda_i, \beta) = \beta \lambda_i x_i^{\beta-1} e^{-\lambda_i x_i^\beta} \quad (2.1)$$

with scale parameter, $\lambda_i > 0$ and shape parameter, $\beta > 0$. We will denote the above model as $(X_i | \lambda_i, \beta) \sim Weib(\lambda_i, \beta)$. The failure rate of the model is given by

$$m_i(x_i | \beta, \lambda_i) = \beta \lambda_i x_i^{\beta-1}, \quad (2.2)$$

where the values $\beta > 1$ (< 1) imply an increasing (decreasing) failure rate distribution, and when $\beta = 1$ the exponential model arises as the special constant failure rate case. The above model implies that the scale parameter, λ_i , depends on the stress environment, but the shape parameter, β , is not, which is a common assumption in the literature [see for example Nelson (1972) and Mazzuchi, Soyer and Vopatek (1997)]. It is common to assume a functional relationship between the failure rate and applied stress level. Such a relationship is referred to as a *time transformation* function. The *Arrhenius law* and the *power law* models are two of the most commonly used time transformation functions [Mann, Shafer and Singpurwalla (1974)].

Assuming a power law model, the relationship between the scale parameter and the applied stress level in the i th accelerated test environment is given by

$$\lambda_i = \theta_1 S_i^{\theta_2}, \quad (2.3)$$

where $\theta_1 > 0$ and θ_2 are unknown coefficients. Alternatively, an Arrhenius law model assumes that

$$\lambda_i = \exp\{\theta_1 - \theta_2/S_i\}. \quad (2.4)$$

In both cases, the aging effect is captured by the shape parameter β , while the effect of the stress environment on the failure behavior is captured by functional parameters θ_1 and θ_2 .

2.1 Bayesian Inference for the ALT Model

Assume that n_i items are tested for t_i units of time at the i th accelerated environment with level S_i . Let X_{ij} denote the time to failure of the j th item tested under the i th environment with realization x_{ij} , $j = 1, 2, \dots, r_i \leq n_i$, where r_i denotes the number of failures observed during the observation period $(0, t_i]$. As noted by Mazzuchi, Soyer and Vopatek (1997), using this notation, we can easily consider the complete sample (no censoring) case, where testing continues until all items have failed, as well as the type I and type II censoring cases. Specifically, the no censoring case is obtained with $t_i = \infty$; type I censoring case is obtained with $t_i < \infty$; and type II censoring case where testing continues until the r th failure is obtained with $t_i = x_{i(r_i)}$, where $x_{i(r_i)}$ is the r th failure time at the i th accelerated environment. In what follows, we will assume that testing is conducted simultaneously at all stress levels, but the case of sequential testing from one stress level to another can also be accommodated.

Let D_i denote the test data from i th accelerated stress environment, that is,

$$D_i = \{n_i, r_i, x_{i1}, \dots, x_{ir_i}\}.$$

Assuming conditional independence of the failure times $\{x_{ij}\}$ given the stress levels $\{S_i\}$, and the parameters β , θ_1 and θ_2 , the likelihood function of λ_i and β from the i th accelerated stress environment is obtained as

$$\mathcal{L}_i(\lambda_i, \beta; D_i) = (\beta\lambda_i)^{r_i} \left\{ \prod_{j=1}^{r_i} (x_{ij})^\beta \right\} \exp\{-\lambda_i T_i(\beta)\}, \quad i = 1, \dots, K, \quad (2.5)$$

where

$$T_i(\beta) = \begin{cases} \sum_{j=1}^{r_i} (x_{ij})^\beta + (n_i - r_i) t_i^\beta & \text{Type I censoring} \\ \sum_{j=1}^{r_i} (x_{ij})^\beta + (n_i - r_i) (x_{i(r_i)})^\beta & \text{Type II censoring} \\ \sum_{j=1}^{n_i} (x_{ij})^\beta & \text{No censoring,} \end{cases}$$

and λ_i is defined either by (2.3) or (2.4). We note that in the case of no censoring $r_i = n_i$ in (2.5). Thus, the likelihood function of β , θ_1 and θ_2 given all the data $D = \{D_1, D_2, \dots, D_K\}$ is

$$\mathcal{L}(\theta_1, \theta_2, \beta; D) = \prod_{i=1}^K \mathcal{L}_i(\lambda_i, \beta; D_i). \quad (2.6)$$

The Bayesian approach requires specification of a prior distribution for β , θ_1 and θ_2 . The joint posterior distribution can then be obtained as

$$p(\theta_1, \theta_2, \beta | D) \propto \mathcal{L}(\theta_1, \theta_2, \beta; D) p(\theta_1, \theta_2, \beta), \quad (2.7)$$

where $p(\theta_1, \theta_2, \beta)$ is the joint prior distribution. As noted by Mazzuchi, Soyer and Vopatek (1997), the posterior distribution in (2.7) can not be obtained in any tractable form for any choice of the prior $p(\theta_1, \theta_2, \beta)$ in (2.7). Thus, in order to perform a fully Bayesian analysis, we will use the Gibbs sampler, a MCMC method which is easy to implement in this case. The Gibbs sampler enables us to generate samples from the posterior distribution $p(\theta_1, \theta_2, \beta | D)$ which can then be used to make assessments about the failure behavior at the use stress environment in our problem. This is achieved without actually computing the distributional form by obtaining successive drawings from the full conditional distributions $p(\theta_1 | \theta_2, \beta, D)$, $p(\theta_2 | \theta_1, \beta, D)$, and $p(\beta | \theta_1, \theta_2, D)$. The process starts with a vector of arbitrary starting values of the parameters, $(\theta_1^{(0)}, \theta_2^{(0)}, \beta^{(0)})$, and continues by iteratively generating samples from the full conditional distributions [see Gelfand and Smith (1990) for more details].

We note that the implementation of the Gibbs sampler requires the availability of the full conditional distributions. If the full conditional distributions are not of any known form or if they do not exist in closed form, some type of random number generation method must be employed to facilitate the implementation of the Gibbs sampler. If the full conditional distributions are logconcave, one can use the efficient adaptive rejection sampling (ARS) method of Gilks and Wild (1992). Unlike standard rejection sampling methods, ARS avoids any maximization steps and therefore can be easily adopted at each iteration of the Gibbs sampler.

In our development for the power law model we will assume a gamma prior for θ_1 , denoted as $Gam[a, b]$, and we will denote the priors for θ_2 and β as $p(\theta_2)$ and $p(\beta)$, respectively. Furthermore, we assume that apriori θ_1 , θ_2 , and β are independent. Under the

power law model, with the given choice of priors, the full conditional distributions are given by

$$p(\theta_1 | \theta_2, \beta, D) \propto \left\{ \theta_1^{\sum_{i=1}^K r_i} \exp \left[-\theta_1 \sum_{i=1}^K S_i^{\theta_2} T_i(\beta) \right] \right\} \theta_1^{a-1} \exp\{-b\theta_1\}, \quad (2.8)$$

$$p(\theta_2 | \theta_1, \beta, D) \propto \left\{ \prod_{i=1}^K S_i^{r_i \theta_2} \right\} \exp \left[-\theta_1 \sum_{i=1}^K S_i^{\theta_2} T_i(\beta) \right] p(\theta_2), \quad (2.9)$$

and

$$p(\beta | \theta_1, \theta_2, D) \propto \beta^{\sum_{i=1}^K r_i} \left\{ \prod_{i=1}^K \left(\prod_{j=1}^{r_i} (x_{ij})^\beta \right) \right\} \exp \left[-\theta_1 \sum_{i=1}^K S_i^{\theta_2} T_i(\beta) \right] p(\beta). \quad (2.10)$$

Note that (2.8) implies $(\theta_1 | \theta_2, \beta, D) \sim \text{Gam} \left[a + \sum_{i=1}^K r_i, b + \sum_{i=1}^K S_i^{\theta_2} T_i(\beta) \right]$, whereas for any reasonable prior form for $p(\theta_2)$ and $p(\beta)$, the posterior conditionals in (2.9) and (2.10) cannot be obtained as a familiar form. However, they are both logconcave if the priors $p(\theta_2)$ and $p(\beta)$ are logconcave since both conditional loglikelihoods are logconcave. Thus the ARS method can be implemented to sample from the posterior densities in (2.9) and (2.10) at each iteration of the Gibbs sampler.

Once a sample $\{\theta_1^{(j)}, \theta_2^{(j)}, \beta^{(j)}\}_{j=1}^J$ is obtained from the posterior distribution $p(\theta_1, \theta_2, \beta | D)$, all the marginal posterior distributions and the posterior moments for the θ_1 , θ_2 and β can be computed from the sample. Given the data D from the K accelerated environments, inferences at the use stress environment S_u can be easily made by using the posterior sample. For example, the predictive reliability at the use stress environment is given by

$$R(x_u | D) = \int R(x_u | \theta_1, \theta_2, \beta) p(\theta_1, \theta_2, \beta | D) d\theta_1 d\theta_2 d\beta, \quad (2.11)$$

where

$$R(x_u | \theta_1, \theta_2, \beta) = e^{-\lambda_u x_u^\beta} \quad (2.12)$$

and λ_u is given by (2.3) or (2.4). The integral in (2.11) can be computed by using a Monte Carlo average using the posterior sample as

$$R(x_u | D) \simeq \frac{1}{J} \sum_{j=1}^J R(x_u | \theta_1^{(j)}, \theta_2^{(j)}, \beta^{(j)}), \quad (2.13)$$

which is the expected reliability at mission time x_u . We can also make probability statements about the reliability at mission time x_u by using a histogram estimate based on $R(x_u | \theta_1^{(j)})$,

$\theta_2^{(j)}, \beta^{(j)}$ $j = 1, \dots, J$. Similarly, expected time to failure at the use stress environment can be obtained. The above setup can be easily modified to incorporate different priors. For example, for the power law model we may assume a lognormal prior for θ_1 and a normal prior for θ_2 .

3. Extensions of the Basic Parametric Model

Earlier work by Blackwell and Singpurwalla (1988) and Mazzuchi and Soyer (1992) considered time transformation functions that change with the stress environment. These authors claimed that such a change is likely to happen due to the changes in the basic failure mechanism with changes in the stress level, and introduced non-Gaussian Kalman filter type models for ALT, and used approximate Bayesian methods for inference.

In what follows we present two models that describe changes in the time transformation function. The first model is a hierarchical Bayes model in the sense of Lindley and Smith (1972) and assumes exchangeability of the parameters of the time transformation function. The second model is a first-order Markov model similar to what is considered by Mazzuchi and Soyer (1992). We note that in both cases the likelihood functions (2.5) and (2.6) will still be valid with changes reflected in λ_i . We focus on the power law model, but extension to the Arrhenius law is straightforward.

3.1 A Hierarchical (Exchangeable) Bayesian Model

To allow for changes in the time transformation function, we rewrite the power law model (2.3) as

$$\lambda_i = \theta_{1i} S_i^{\theta_2}, \quad (3.1)$$

where the parameter θ_{1i} depends on the stress as reflected by the index i . Assume θ_{1i} 's are conditionally independent with $\theta_{1i} \sim Gam[a, b]$, where a and b are random quantities with *hyperprior* $p(a, b)$. This implies that $\{\theta_{1i}\}$, $i = 1, \dots, K$, is an exchangeable sequence. If we further assume a priori independence of β and θ_2 , and of θ_{1i} 's, we can recast the ALT model into a hierarchical Bayesian model as follows:

$$\begin{aligned} (X_i | \lambda_i, \beta) &\sim Weib(\lambda_i, \beta) \\ \lambda_i &= \theta_{1i} S_i^{\theta_2} \\ (\theta_2, \beta) &\sim p(\theta_2) p(\beta) \\ (\theta_{1i} | a, b) &\sim Gam[a, b] \\ (a, b) &\sim p(a, b). \end{aligned} \quad (3.2)$$

For any choice of priors in (3.2), the posterior inference cannot be performed analytically and thus we use a Gibbs sampler. The full conditionals for $\{\theta_{1i}\}$ are given by

$$(\theta_{1i}|a, b, \theta_1^{(-i)}, \theta_2, \beta, D) \sim \text{Gam}\left[a + r_i, b + S_i^{\theta_2} T_i(\beta)\right], \quad (3.3)$$

where $\theta_1^{(-i)} = \{\theta_{1j}; j \neq i\}$. Assuming $p(a, b) = p(a) p(b)$, and $b \sim \text{Gam}[c, d]$, the full conditional for b obtains as

$$(b|\boldsymbol{\theta}_1, \theta_2, \beta, a, D) \sim \text{Gam}\left[c + aK, d + \sum_{i=1}^K \theta_{1i}\right], \quad (3.4)$$

where $\boldsymbol{\theta}_1 = (\theta_{1i}; i = 1, \dots, K)$. Similarly, the full conditional of a is

$$p(a|\boldsymbol{\theta}_1, \theta_2, \beta, b, D) \propto \left[\frac{b^a}{\Gamma(a)} \prod_{i=1}^K (\theta_{1i})^{a-1}\right] p(a) \quad (3.5)$$

which is log-concave as long as $p(a)$ is a log-concave density.

For θ_2 , we have

$$p(\theta_2|\boldsymbol{\theta}_1, \beta, a, b, D) \propto \left\{ \prod_{i=1}^K S_i^{r_i \theta_2} \right\} \exp\left[-\sum_{i=1}^K \theta_{1i} S_i^{\theta_2} T_i(\beta)\right] p(\theta_2),$$

which is log-concave for a log-concave prior. Finally, for β , we get

$$p(\beta|\boldsymbol{\theta}_1, \theta_2, a, b, D) \propto \beta^{\sum_{i=1}^K r_i} \left\{ \prod_{i=1}^K \left(\prod_{j=1}^{r_i} (x_{ij})^\beta \right) \right\} \exp\left[-\sum_{i=1}^K \theta_{1i} S_i^{\theta_2} T_i(\beta)\right] p(\beta),$$

which is also log-concave for a log-concave prior $p(\beta)$. Thus, we can use the ARS method to sample from the conditionals for θ_2 and β .

Having obtained the posterior samples, we focus on the predictive reliability at the use-stress

$$R(x_u|D) = \int R(x_u|\theta_{1u}, \theta_2, \beta) p(\theta_{1u}, \theta_2, \beta|D) d\theta_{1u} d\theta_2 d\beta. \quad (3.6)$$

Given the posterior samples we can approximate (3.6) as

$$R(x_u|D) \simeq \frac{1}{J} \sum_{j=1}^J R(x_u|\theta_{1u}^{(j)}, \theta_2^{(j)}, \beta^{(j)}), \quad (3.7)$$

where $\theta_{1u}^{(j)}$ is generated from the gamma density $p(\theta_{1u}|a^{(j)}, b^{(j)})$, conditional on the posterior realizations $(a^{(j)}, b^{(j)})$.

3.2 A Markov (Dynamic) Model

An alternate modeling strategy for capturing changes in the time transformation function is to consider a first-order Markov structure on $\alpha_i = \ln \theta_{1i}$ in (3.1). Since S_i and S_{i-1} are ordered neighboring stress environments $S_i < S_{i-1}$, for $i = 1, \dots, K$, we assume that

$$(\alpha_i | \alpha_{i-1}, \phi, \tau) \sim N(\phi \alpha_{i-1}, \tau^{-1}), \quad (3.8)$$

where ϕ and τ are unknown quantities. Initially, we assume that $\alpha_0 \sim N(\mu_0, \tau^{-1})$. This is a generalization of the model in Mazzuchi and Soyer (1992). Note that (3.8) can be represented as a first-order autoregressive AR(1) dynamic model as

$$\alpha_i = \phi \alpha_{i-1} + \epsilon_i, \quad \epsilon_i \sim N(0, \tau^{-1}). \quad (3.9)$$

The use-stress can be considered as the $(K+1)$ st level in the above setup, that is, $\alpha_u = \alpha_{K+1}$. In cases where the stresses are not equally spaced the precision τ can be discounted by a function $f(S_i, S_{i-1})$ as $\tau_i = \tau f(S_i, S_{i-1})$. A suitable choice for f is $f(S_i, S_{i-1}) = S_i/S_{i-1}$ where $S_i < S_{i-1}$.

Specification of the Markov model can be completed by assuming independent priors as $\tau \sim \text{Gam}(a, b)$ and $p(\phi)$ in the above. Although $-1 < \phi < 1$ would imply stationarity of (3.9), it is not strictly necessary in our analysis, so we choose a normal prior for ϕ in convenience; $\phi \sim N(\mu_\phi, \tau_\phi^{-1})$. Finally, we assume independent priors on β and θ_2 . Thus, for $0 < i < K$, the full conditional of α_i is given by

$$\begin{aligned} & p(\alpha_i | \alpha^{(-i)}, \theta_2, \phi, \beta, \tau, D) \\ & \propto \exp \left[\alpha_i r_i - e^{\alpha_i} S_i^{\theta_2} T_i(\beta) - \frac{\tau}{2} \left((\alpha_i - \phi \alpha_{i-1})^2 + (\alpha_{i+1} - \phi \alpha_i)^2 \right) \right], \end{aligned} \quad (3.10)$$

which is log-concave. For $i = K$, that is, for the smallest stress environment the full conditional is obtained as

$$\begin{aligned} & p(\alpha_K | \alpha^{(-K)}, \theta_2, \phi, \beta, \tau, D) \\ & \propto \exp \left[\alpha_K r_K - e^{\alpha_K} S_K^{\theta_2} T_K(\beta) - \frac{\tau}{2} (\alpha_K - \phi \alpha_{K-1})^2 \right], \end{aligned} \quad (3.11)$$

which is also a log-concave density. For α_0 it can be shown that

$$(\alpha_0 | \boldsymbol{\alpha}, \theta_2, \phi, \beta, \tau, D) \sim N \left[\frac{\mu_0 + \phi \alpha_1}{1 + \phi^2}, (\tau(1 + \phi^2))^{-1} \right], \quad (3.12)$$

where $\boldsymbol{\alpha} = \{\alpha_i; i = 1, \dots, K\}$, and for precision τ

$$(\tau | \alpha_0, \boldsymbol{\alpha}, \theta_2, \phi, \beta, D) \sim \text{Gam}(a^*, b^*),$$

with parameters $a^* = a + (K + 1)/2$ and $b^* = b + \frac{1}{2} [(\alpha_0 - \mu_0)^2 + \sum_{i=1}^K (\alpha_i - \phi \alpha_{i-1})^2]$. The full conditional of ϕ is

$$p(\phi | \alpha_0, \boldsymbol{\alpha}, \theta_2, \tau, \beta, D) \propto \exp \left[-\frac{\tau}{2} \sum_{i=1}^K (\alpha_i - \phi \alpha_{i-1})^2 \right] p(\phi).$$

If the prior on ϕ is normal, that is, $\phi \sim N(\mu_\phi, \tau_\phi^{-1})$, then the full conditional of ϕ will be a normal given by

$$(\phi | \alpha_0, \boldsymbol{\alpha}, \theta_2, \tau, \beta, D) \sim N \left[\frac{\tau_\phi \mu_\phi + \tau \sum_{i=1}^K (\alpha_i \alpha_{i-1})}{\tau_\phi + \tau \sum_{i=1}^K \alpha_{i-1}^2}, \left(\tau_\phi + \tau \sum_{i=1}^K \alpha_{i-1}^2 \right)^{-1} \right]. \quad (3.13)$$

For any other log-concave prior the full conditional of ϕ is not available in closed form but is log-concave.

The full conditionals of θ_2 and β are log-concave and are given by

$$p(\theta_2 | \alpha_0, \boldsymbol{\alpha}, \phi, \tau, \beta, D) \propto \left\{ \prod_{i=1}^K S_i^{r_i \theta_2} \right\} \exp \left[-\sum_{i=1}^K e^{\alpha_i} S_i^{\theta_2} T_i(\beta) \right] p(\theta_2)$$

and

$$p(\beta | \alpha_0, \boldsymbol{\alpha}, \theta_2, \tau, \phi, D) \propto \beta^{\sum_{i=1}^K r_i} \left\{ \prod_{i=1}^K \left(\prod_{j=1}^{r_i} (x_{ij})^\beta \right) \right\} \exp \left[-\sum_{i=1}^K e^{\alpha_i} S_i^{\theta_2} T_i(\beta) \right] p(\beta).$$

Once the posterior analysis is completed we can obtain the predictive reliability at the use-stress using (3.7) where $\theta_{1u} = \exp(\alpha_u)$ and $\alpha_u = \alpha_{K+1}$. Given the posterior samples we can approximate the predictive reliability via (3.7) with $\theta_{1u}^{(j)} = \alpha_u^{(j)}$ and $\alpha^{(j)}$ is generated from the normal density $p(\alpha_u | \alpha_K^{(j)}, \phi^{(j)}, \tau^{(j)})$ for each posterior realization $(\alpha_K^{(j)}, \phi^{(j)}, \tau^{(j)})$.

4. Analysis of Accelerated Life Testing Data

In this section we illustrate the Bayesian analysis of the presented models using two accelerated life-testing data sets, investigate the appropriateness of dynamic time transformation functions and discuss model comparison. In so doing, we first discuss the deviance information criterion for comparing our models.

4.1 Model Comparison

In comparing the performance of basic parametric model with hierarchical and Markov extensions presented above, computation of the Bayes factors [see Kass and Raftery (1995) for a comprehensive review] is difficult. The marginal likelihoods for the competing models, which are needed to compute the Bayes factor, cannot be directly approximated from the Gibbs sampler,

An alternative approach is to use a model selection criterion such as the *Deviance Information Criterion (DIC)* of Spiegelhalter et. al. (2002). For a generic parameter vector Θ , *DIC* is defined as

$$DIC = \bar{D} + p_D, \quad (4.1)$$

where $D = -2\log\mathcal{L}(\Theta)$, is two times the negative loglikelihood, $\bar{D} = E_{\Theta|data}[D]$ and $p_D = \bar{D} - D(\hat{\Theta})$, where $\hat{\Theta}$ is the posterior mean. The *DIC* has the general "fit + complexity" form used by many model selection criteria. In (4.1) \bar{D} represents the "goodness of the fit of the model where p_D represents a complexity penalty as reflected by the effective number of parameters of the model.

4.2 Example: Breakdowns of an Insulating Fluid

In what follows, we will illustrate the use of our methodology to the accelerated life test data published in Nelson (1972). The data is given in Table 1 and represents the times to breakdown of an insulating fluid subjected to various voltage levels. The accelerated stress levels are given by 26, 28, 30, 32, 34, 36, and 38 Kv and we are interested in making inference for the breakdown times for the insulating fluid at the use stress of 22 Kv. Following the analysis of Nelson (1972) and Mazzuchi, Soyer and Vopatek (1997), we will assume a power law model for the time transformation function for the data.

We will consider three models to analyze these data, the basic parametric model of Section 2, the hierarchical Bayesian model of Section 3.1, and the Markov model of Section 3.2. Comparison among them will be made based on the *DIC*. Due to our unfamiliarity with the problem, we will use diffuse priors in our analysis. In all models we choose a uniform prior in the range (0, 10) for the shape parameter β and a uniform prior in the range (0, 100) for θ_2 . In the parametric model of Section 2, the prior for θ_1 is *Gam*[0.01, 0.01]. In the hierarchical Bayes model we use independent gamma priors for a and b in (3.2) with the parameters 0.01 and 0.01. In the Markov model the prior for ϕ is uniform over $(-1, 1)$ and for τ is *Gam*[0.01,0.01].

A comparison of the four models using *DIC* is given in Table 2. We note that the *DIC*'s for the three models are all very similar with the parametric model having the lowest

DIC. It is interesting to note that the hierarchical and Markov models estimate p_D , the effective number of parameters, as 4 or 5. Thus, the data do not seem to support the hypothesis that the time transformation function varies with the stress level.

TABLE 1
Times to Breakdown of an Insulating Fluid (in Minutes)
Under Various Values of the Stress

38Kv	36Kv	34Kv	32Kv	30Kv	28Kv	26Kv
.09	.35	.19	.27	7.74	68.85	5.79
.39	.59	.78	.40	17.05	108.29	1579.52
.47	.96	.96	.69	20.46	110.59	2323.70
.73	.99	1.31	.79	21.02	426.07	-
.74	1.69	2.78	2.75	22.66	1067.6	-
1.13	1.97	3.16	3.91	43.40	-	-
1.40	2.07	4.15	9.88	47.30	-	-
2.38	2.59	4.67	13.95	139.07	-	-
-	2.71	4.85	15.93	141.12	-	-
-	2.90	6.50	27.80	175.88	-	-
-	3.67	7.35	53.24	194.90	-	-
-	3.99	8.01	82.85	-	-	-
-	5.35	8.27	89.29	-	-	-
-	13.77	12.06	100.58	-	-	-
-	25.50	31.75	215.10	-	-	-
-	-	32.52	-	-	-	-
-	-	33.91	-	-	-	-
-	-	36.71	-	-	-	-
-	-	72.89	-	-	-	-

This conclusion is also supported by Figure 1 where we present the posterior distribution θ_1 under the parametric model (denoted by P) and the posterior distributions of $\theta_{11}, \dots, \theta_{17}$ under the hierachical model. We note that the posterior distributions of θ_{1j} 's do not change much from one stress level to the other.

TABLE 2
Model Comparison Using DIC

Model	<i>DIC</i>	p_D
Parametric Model	607.85	3.33
Hierarchical Bayes Model	609.77	4.57
Markov Model	609.66	4.66

Posterior Distributions of theta1 and theta1[1:7]

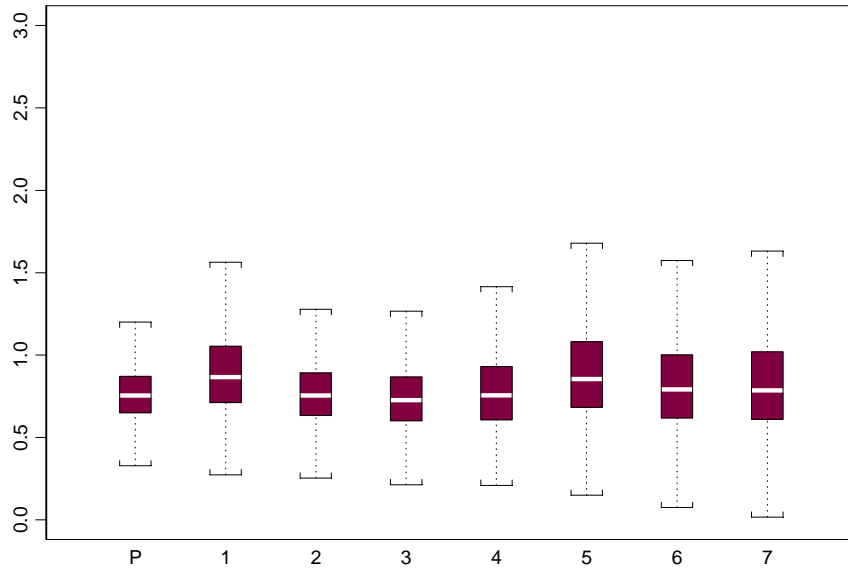


Figure 1. Comparison of Posterior Distributions of θ_1 and θ_{1j} 's.

In Figure 2, we present the marginal posterior distributions of β , θ_1 and θ_2 under the basic model. The posterior distribution of β is concentrated around 0.8 implying a decreasing failure rate at any stress level. The distribution of parameter θ_2 is concentrated around high positive values implying that failure rate increases with stress level as expected.

Posterior Distributions under the Parametric Model

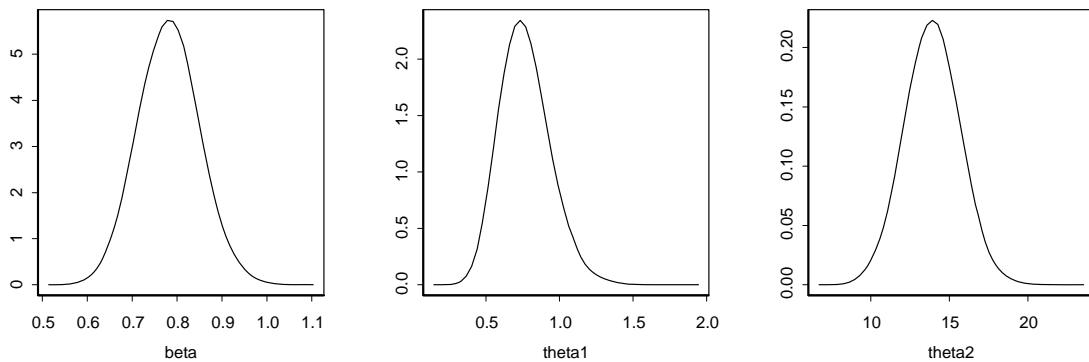


Figure 2. Posterior Distributions of β , θ_1 and θ_2 in the Parametric Model.

The predictive reliability function for the use stress $S_u = 22\text{Kv}$ is given in Figure 3 with 0.05 and 0.95 reliability bounds. We note that the items are quite reliable at the use stress level as implied by high reliability values for $X_u < 10,000$ which is expected in ALTs.

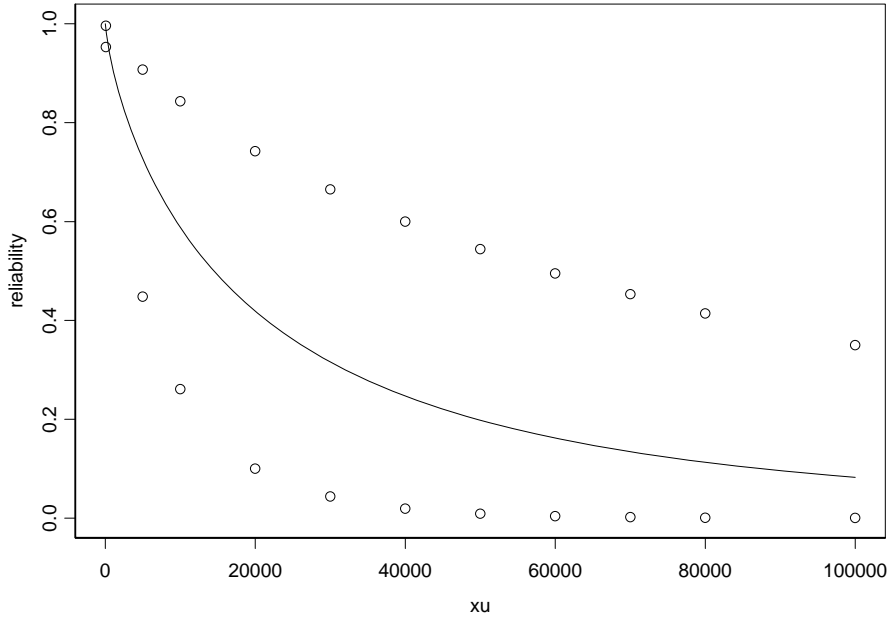


Figure 3. Predictive Reliability Function at $S_u = 22\text{ Kv}$.

4.3 Example: Rolling Bearing Data

The second data set we consider is taken from Nelson (1990, pp. 156) and it represents complete life data from a load accelerated life test of rolling bearings. Four test loads were used and ten bearings were tested at each of the test loads. The author suggested a Weibull model with a power law relationship. Life data is given in 10^6 revolutions. The data was analyzed using the three models and a power law time transformation function as in Section 4.2.

A comparison of the three models using DIC is given in Table 4. Note that the DIC 's under hierarchical and Markov models are considerably lower than that of the basic parametric model. The estimated effective number of parameters, p_D , is given as 6 under both of these models implying that there are four different θ_{1j} 's. Thus, there is evidence that the time transformation function changes with stress level in this data. We can also see this by looking at the posterior distributions of θ_{1j} 's in the hierarchical model. These are shown in Figure 4 and compared to the distribution of θ_1 in the parametric model (denoted by P in Figure 4). The differences in the distributions are easily notable in this case. Similar results are

obtained for the Markov model. Posterior distributions of the hyperparameters a and b are presented in Figure 5. We note that the posterior distributions for both of these parameters are very peaked even though the priors were flat.

TABLE 3
Times to Failure of Roll Bearings (in 10^6 Revolutions)
Under Various Loads

0.87 Ld.	0.99 Ld.	1.09 Ld.	1.18 Ld.
1.67	.80	.012	.073
2.20	1.00	.18	.098
2.51	1.37	.20	.117
3.00	2.25	.24	.135
3.90	2.95	.26	.175
4.70	3.70	.32	.262
7.53	6.07	.32	.27
14.70	6.65	.42	.35
22.76	7.05	.44	.386
37.40	7.37	.88	.456

TABLE 4
Model Comparison Using DIC

Model	DIC	p_D
Parametric Model	114.82	3.02
Hierarchical Bayes Model	109.08	5.59
Markov Model	109.43	6.03

The posterior distributions of β and θ_2 for the hierarchical Bayes model are shown in Figure 6. The posterior distribution of β is concentrated around values greater than 1 implying increasing failure rate for the items at a given load. The posterior distribution of θ_2 is peaked around 20 showing a sharp increase in failure rate with increasing load.

In view of the above, the claim that the time transformation function may change with the stress environment is supported by some accelerated life test data. Based on our analysis it seems that both the hierarchical and dynamic models are able to capture such changes.

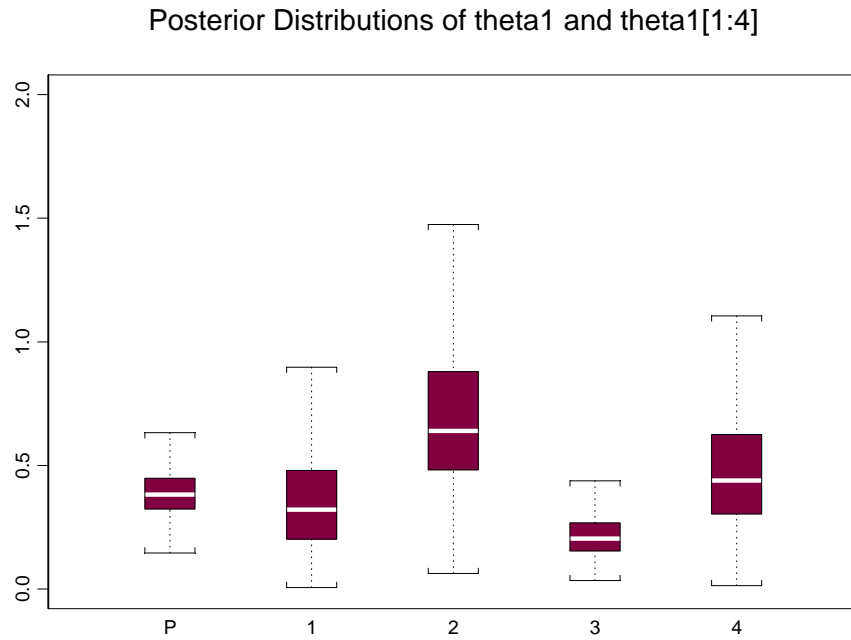


Figure 4. Posterior Distributions of θ_1 and θ_{1j} 's under the Hierarchical Model.

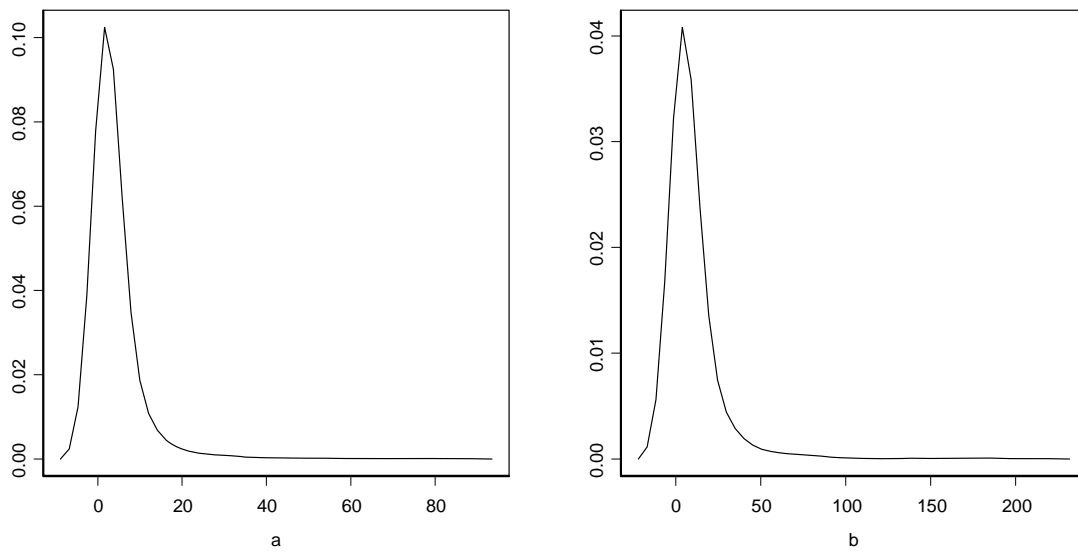


Figure 5. Posterior Distributions of a and b under the Hierarchical Model.

5. Concluding Remarks

In this paper we presented Bayesian analyses of two classes of parametric models for accelerated life tests where the time transformation function changes with the stress environment. Our results show that such models may be more appropriate than the static models in some accelerated life tests. In our setup, we considered a power law model for the time transformation function and assumed a Weibull failure model. The approach can be easily extended to other time transformation functions and failure models. Alternatively, a semiparametric generalization of the models can be considered by relaxing the parametric assumptions on the distributions of the parameters of the time transformation function. Although semiparametric models have been used in Bayesian survival analysis, see for example, Kuo and Mallick (1997), consideration of these models are new in ALT literature. The semiparametric ALT models uses mixtures of Dirichlet processes and inference for these involves use of MCMC methods. Such work is presently under consideration.

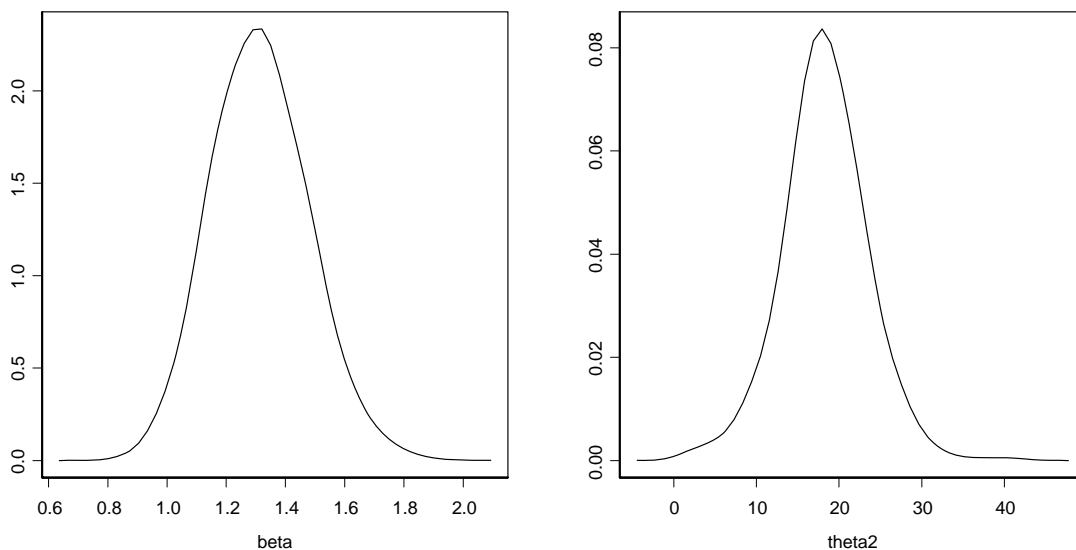


Figure 6. Posterior Distributions of β and θ_2 under the Hierarchical Model.

References

Blackwell, L. M. and Singpurwalla, N. D. (1988). Inference from accelerated life tests using filtering in colored noise. *Journal of the Royal Statistical Society, Series B*, Vol. 50, pp. 281-292.

- Gelfand, A. E. and Smith, A. F. M. (1990). Sampling-based approaches to calculating marginal densities, *Journal of the American Statistical Association*, Vol. 85, pp. 398-409.
- Gilks, W. R. and Wild, P. (1992). Adaptive rejection sampling for Gibbs sampling, *Applied Statistics*, Vol. 41, pp. 337-348.
- Kass, R. E. and Raftery, A. E. (1995). Bayes factors. *Journal of the American Statistical Association*, Vol. 90, pp.773-777.
- Kuo, L. and Mallick, B. K. (1997). Bayesian semi-parametric inference for the accelerated failure time model, *Canadian Journal of Statistics*, Vol. 25, pp. 457-472.
- Lindley, D. V., and Smith, A. F. M. (1972). Bayes estimates for the linear model (with discussion). *Journal of the Royal Statistical Society, Ser. B*, Vol. 34, pp. 1-41.
- Mann, N. R., Schafer, R. E. and Singpurwalla, N. D. (1974). *Methods for Statistical Analysis of Reliability and Life Data*, John Wiley and Sons, Inc., New York.
- Mazzuchi, T. A. and Soyer, R. (1992). A dynamic general linear model for inference from accelerated life testing, *Naval Research Logistics Quarterly*, 39, pp. 757-773.
- Mazzuchi, T. A., Soyer, R and Vopatek, A. L. (1997). Linear Bayesian inference for accelerated Weibull model, *LifeTime Data Analysis* , Vol. 3, pp. 1-13.
- Nelson, W. B. (1972). Graphical analysis of accelerated life test data with the inverse power law model, *IEEE Transactions on Reliability*, R-21, pp. 2-11.
- Nelson, W. B. (1990). *Accelerated Life Testing*, Wiley, New York.
- Spiegelhalter, D. J., Best, N. G, Carlin, B. P. and van der Linde, A. (2002). Bayesian measures of model complexity and fit, *Journal of the Royal Statistical Society, Series B*, Vol. 64, pp. 1-34.
- van Dorp, R., Mazzuchi, T. A., Fornell, G. and Pollock, L. R. (1996). A Bayes approach to step-stress accelerated life testing, *IEEE Transactions on Reliability*, R-45, pp. 491- 498.
- van Dorp, R., Mazzuchi, T. A. (2004a). A General Bayes Weibull Inference Model for Accelerated Life Testing, *Reliability Engineering and System Safety*, Vol. 90, pp. 140-147.
- van Dorp, R., Mazzuchi, T. A. (2004b). A General Bayes exponential Inference Model for Accelerated Life Testing, *Jrnl. of Statist. Plan. & Inf.*, Vol. 119, pp. 55-74.