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Assessment of Mortgage Default Risk via Bayesian State-Space Models

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September 17, 2011

Abstract

Managing risk at the aggregate level is crucial for banks and financial institutions as required by the Basel II framework. In modeling aggregate mortgage default rate, assessing if the rate exhibits a dynamic behavior and identifying effect of macroeconomic variables on the default rate are issues of concern to both practitioners and researchers. In addressing these issues, we introduce discrete time Bayesian state space models with Poisson measurements to model aggregate mortgage default rate. In doing so we discuss parameter updating and estimation using Markov chain Monte Carlo methods. In assessing the dynamic nature of the mortgage default rate, we compare the forecasting performance of the proposed models with a Bayesian Poisson regression model used as a benchmark. We illustrate the use of the proposed models using actual U.S. residential mortgage data and discuss insights gained from Bayesian analysis.

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1 Introduction

Given the large size of outstanding residential mortgage loans in the U.S., a healthy mortgage market is important for stability of the financial markets and the whole economy. Due to its significant costs upon mortgage borrowers, lenders, insurers and investors of mortgage backed securities, management of mortgage default risk is one of the primary concerns for the policy makers and financial institutions.

Most commonly used measures of mortgage default risk are delinquency and foreclosure rates of mortgage loans. They provide a general description of how the mortgage market perform, compared to the macro economy. According to Gilberto and Houston (1989), mortgage default is legally defined as the transfer of property ownership from the borrower to the lender. Majority of researchers who focus on modeling of default risk define mortgage default as being delinquent in a mortgage payment for 90 days as discussed in Ambrose and Capone (1998). In this paper we use the latter definition to distinguish default from foreclosure.

Most of the work in the mortgage default risk literature has focused mainly on the individual default behavior of borrowers, and the effects of mortgage loan, property, borrower and economic characteristics on default risk. Quercia and Stegman (1992) provide a detailed literature review of research in mortgage default risk until 1992. More recent developments can be found in Leece (2004). There are two dominant classes of models in the literature. The first class of models is based on the ruthless default assumption and is option theoretic where the mortgage value, prepayment and default options are determined via stochastic behavior of prices and interest rates as in Kau et al. (1990). The second class is based on the hazard rate models where time to mortgage default is a random variable with hazard rate as a function of individual borrower and loan characteristics as studied by Lambrecht et al. (1997), Lambrecht et al. (2003) and Soyer and Xu (2010). Both classes of models are based on the behavior of individual mortgages. But studying the default behavior at the aggregate level is also of interest to financial institutions and policy makers to be able to predict default rates and to develop appropriate mitigation instruments. As pointed out by Taufer (2006), managing risk at the aggregate level is crucial for banks and financial institutions as required by the Basel II framework which encourages banks to identify and manage present and future risks. Taufer (2006) models the probability of default at the aggregate level for two default classes, all-corporate and speculative-grades in U.S. as a stochastic process.

Modeling aggregate default rates requires consideration of several issues. First, it is important to identify the effect of macroeconomic variables on the aggregate default rate. This is pointed out by Taufer (2006) but is not considered in his model. Another issue to assess if the aggregate default rate exhibits a dynamic behavior. In modeling individual default rates, Soyer and Xu (2010) point out that default rates are nonmonotonic. More specifically, the authors report that default rates are typically first increasing and then decreasing over the duration of the mortgage. It is not unreasonable to expect that the aggregate default rate will also follow such a dynamic behavior. Thirdly, as noted by Kiefer (2011), it is not uncommon to have correlated defaults over time. Thus, it is desirable for models to capture such correlations.

In this paper, we present a discrete time Bayesian state-space model for Poisson counts to address the above issues. The proposed model enables us to describe the dynamic behavior of aggregate mortgage default rates over time and assumes a Markovian structure to describe the correlated default rates. This Markovian structure enables us to capture correlations between the number of defaults over time and provides an alternate way of modeling time-series of counts. Since the Markovian structure is assumed for the parameter, that is, for the default rate, our model can be classified as a *parameter driven Markov model* using the terminology of Cox (1981). We introduce an extension of the model by modulating the default rate by considering the effect of covariates describing the economic environment. This model can be considered as a discrete time version of *modulated Poisson process model* of Cox (1972). This class of models and their Bayesian analysis have not been considered in the literature before. To the best of our knowledge only a few studies consider Bayesian methods in modeling mortgage default risk in the literature. Herzog (1988) introduces basic Bayesian concepts and Popova et al. (2008) apply Bayesian methods to forecast mortgage prepayment rates. More recently, Kiefer (2010) introduces the incorporation of expert knowledge in estimating default rates from a Bayesian point of view, detail a binomial model with dependent defaults and discuss implications of such models on risk management. As noted by Kiefer (2011) the Bayesian approach provides a coherent framework to combine data with prior information and enables us make inferences using probabilistic reasoning. As will be discussed in our illustrations additional insights are gained from the Bayesian analysis.

A summary of our paper is as follows: In Section 2, we introduce a Bayesian state space model for the monthly default counts for a given mortgage pool. Section 3 is dedicated to the extension of

the discrete time Bayesian state space model with covariates. We discuss the Bayesian analysis of the models in Section 4 using Markov chain Monte Carlo methods. An illustration of the proposed models is presented in Section 5 using real default count data for different mortgage pools where we discuss both in and out of sample fit issues for our models and compare them with the Bayesian Poisson regression which we use as a benchmark. Finally, in Section 6 we conclude with a summary of our findings and suggestions for future work.

2 A Dynamic Model for Number of Defaults

In what follows, we first introduce a discrete time Bayesian model with Poisson observations and a default rate that evolves over time according to a Markov process. This model does not take into account the effects of covariates on the default rate of a given mortgage pool. Smith and Miller (1986) consider a similar state space model for exponential measurements which was used by Morali and Soyer (2003) in the context of software reliability.

Let N_t be the number of defaults of a given mortgage pool during the month t and θ_t be its default rate. Given θ_t, we assume that the number of defaults during the tth month is described via a non-homogeneous Poisson process,

$$(N_t|\theta_t) \sim Pois(\theta_t). \tag{2.1}$$

In (2.1) it is assumed that that given the default rate θ_t, the default counts, N_ts are conditionally independent. We note that (2.1) acts as an observation equation for discrete time.

For the state evolution equation of θ_ts, we assume that consecutive default rates exhibit a Markovian behavior similar to that considered by Taufer (2006) at the aggregate level. The Markovian behavior of default rates is described by

$$\theta_t = \frac{\theta_{t-1}}{\gamma}\epsilon_t, \tag{2.2}$$

where $(\epsilon_t|N^{(t-1)}) \sim Beta[\gamma a_{t-1}, (1-\gamma)a_{t-1}]$ with $a_{t-1} > 0$, $0 < \gamma < 1$, and $N^{(t-1)} = \{N_1, \cdots, N_{t-1}\}$. Here, γ acts like a discounting term between consecutive default rates. The state equation (2.2)

implies a stochastic ordering between the default rates, $\theta_t < \frac{\theta_{t-1}}{\gamma}$. Therefore, it can be shown that

$$(\theta_t|\theta_{t-1}, N^{(t-1)}) \sim Beta[\gamma a_{t-1}, (1-\gamma)a_{t-1}; (0, \frac{\theta_{t-1}}{\gamma})], \qquad (2.3)$$

that is, a scaled Beta density. If one assumes that a priori θ_0 follow a gamma density as

$$(\theta_0|N^{(0)}) \sim Gamma(a_0, b_0), \qquad (2.4)$$

then one can develop an analytically tractable Bayesian analysis for the model. Following Smith and Miller (1986), as a result of (2.2) and (2.4) we can obtain

$$(\theta_{t-1}|N^{(t-1)}) \sim Gamma(a_{t-1}, b_{t-1}), \qquad (2.5)$$

which can be shown by induction. Given the measurement equation (2.1), the state evolution equation (2.2) and the prior (2.4), the posterior default rates and one-step-ahead default count densities can be obtained analytically.

Predictive density for the default rate given default counts up to time $t-1$ is given by

$$(\theta_t|N^{(t-1)}) \sim Gamma(\gamma a_{t-1}, \gamma b_{t-1}). \qquad (2.6)$$

It follows from the above that $E(\theta_t|N^{(t-1)}) = E(\theta_{t-1}|N^{(t-1)})$, whereas $V(\theta_t|N^{(t-1)}) = \frac{V(\theta_{t-1}|N^{(t-1)})}{\gamma}$. In other words, the model implies that as we move forward in time, expected default rate stays the same but our uncertainty about the rate increases.

The posterior density of the default rate given default counts up to time t is given by

$$(\theta_t|N^{(t)}) \sim Gamma(a_t, b_t), \qquad (2.7)$$

where $a_t = \gamma a_{t-1} + N_t$ and $b_t = \gamma b_{t-1} + 1$. The posterior density (2.7) is also known as the filtering distribution of the default rate.

Finally, one month ahead forecasting density for N_t given the default counts up to month $t-1$ can be obtained via

$$(N_t|N^{(t-1)}) \sim Negbin(r_t, p_t), \qquad (2.8)$$

where $r_t = \gamma a_{t-1}$ and $p_t = \frac{\gamma b_{t-1}}{\gamma b_{t-1}+1}$. As summarized above, conditional on the discount factor γ, the updating of the default rate in the light of new default information and one month ahead forecasting densities for default counts are all available analytically. Another attractive feature of the proposed model is that in addition to obtaining point estimates of the default counts and the default rates at each point in time, one can also obtain well known probability distributions with easy to obtain statistical properties such as the mode, median, standard deviation and credibility intervals.

3 Dynamic Model with Covariates

We next extend the model of Section 2 by considering the effects of covariates on the dynamic default rate. Let N_t be the number of defaults of a given mortgage pool during the month t and λ_t be its default rate. We assume that the default rate is given by

$$\lambda_t = \theta_t e^{\boldsymbol{\beta}' \boldsymbol{z_t}}, \tag{3.1}$$

where $\boldsymbol{z_t}$ is the vector of the covariates and $\boldsymbol{\beta}$ is the parameter vector. θ_t acts like the baseline default rate which evolves over time. Given λ_t, we assume that the number of defaults during the tth month is described via a modulated non-homogeneous Poisson process,

$$(N_t|\theta_t, \boldsymbol{\beta}, \boldsymbol{z_t}) \sim Pois(\theta_t e^{\boldsymbol{\beta}' \boldsymbol{z_t}}). \tag{3.2}$$

The modulated Poisson model (3.2) acts as an observation equation defined over discrete time. For the state evolution equation of the baseline failure rate, θ_t we assume a similar structure as before given by (2.2). In addition, we assume that initially $(\theta_0|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(0)}) \sim Gamma(a_0, b_0)$ and is independent of $\boldsymbol{\beta}$. Thus, it can be shown that the conditional distribution of $(\theta_{t-1}|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)})$ follows a gamma density as

$$(\theta_{t-1}|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)}) \sim Gamma(a_{t-1}, b_{t-1}). \tag{3.3}$$

Therefore, the conditional posterior density of θ_t given $\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)}$ can be obtained via

$$p(\theta_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)}) = \int_{\gamma\theta_t}^{\infty} p(\theta_t|\theta_{t-1}, N^{(t-1)})p(\theta_{t-1}|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)})d\theta_{t-1}, \tag{3.4}$$

which reduces to a gamma density as

$$(\theta_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)}) \sim Gamma(\gamma a_{t-1}, \gamma b_{t-1}). \tag{3.5}$$

Furthermore, the conditional posterior of θ_t given $\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t)}$ can be obtained using (3.2) and (3.5) and the Bayes Rule

$$p(\theta_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t)}) \propto p(N_t|\boldsymbol{\beta}, \boldsymbol{z_t}, \theta_t)p(\theta_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)}). \tag{3.6}$$

The above implies that

$$p(\theta_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t)}) \propto (\theta_t e^{\boldsymbol{\beta'z_t}})^{\gamma a_{t-1}+N_t-1}e^{-(\gamma b_{t-1}+1)(\theta_t e^{\boldsymbol{\beta'z_t}})},$$

that is, the conditional distribution of the default rate at time t is a gamma density given by

$$(\theta_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t)}) \sim Gamma(a_t, b_t), \tag{3.7}$$

where $a_t = \gamma a_{t-1} + N_t$ and $b_t = \gamma b_{t-1} + e^{\boldsymbol{\beta'z_t}}$.

The one-step ahead conditional predictive distribution of default counts at time t given $\boldsymbol{\beta}, \boldsymbol{z_t}$ and $N^{(t-1)}$ can be obtained via

$$p(N_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)}) = \int_0^{\infty} p(N_t|\boldsymbol{\beta}, \boldsymbol{z_t}, \theta_t)p(\theta_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)})d\theta_t, \tag{3.8}$$

where $(N_t|\boldsymbol{\beta}, \boldsymbol{z_t}, \theta_t) \sim Pois(\theta_t e^{\boldsymbol{\beta'z_t}})$ and $(\theta_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)}) \sim Gamma(\gamma a_{t-1}, \gamma b_{t-1})$. Therefore,

$$p(N_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)}) = \binom{\gamma a_{t-1} + N_t - 1}{N_t}\{\frac{\gamma b_{t-1}}{\gamma b_{t-1} + e^{\boldsymbol{\beta'z_t}}}\}^{\gamma a_{t-1}}\{\frac{e^{\boldsymbol{\beta'z_t}}}{\gamma b_{t-1} + e^{\boldsymbol{\beta'z_t}}}\}^{N_t}, \tag{3.9}$$

which is a negative binomial model denoted as

$$(N_t | N^{(t-1)}, \boldsymbol{\beta}, \boldsymbol{z_t}) \sim Negbin(r_t, p_t), \tag{3.10}$$

where $r_t = \gamma a_{t-1}$ and $p_t = \frac{\gamma b_{t-1}}{\gamma b_{t-1} + e^{\boldsymbol{\beta}' \boldsymbol{z_t}}}$. The predictive density (3.10) implies that given the covariates and the default counts up to month $t-1$, forecasts for the month t is a function of the observed default count in month $t-1$ adjusted by the corresponding covariates. The mean of $(N_t | N^{(t-1)}, \boldsymbol{\beta}, \boldsymbol{z_t})$, can be computed via

$$E(N_t | N^{(t-1)}, \boldsymbol{\beta}, \boldsymbol{z_t}) = \frac{a_{t-1}}{b_{t-1}} e^{\boldsymbol{\beta}' \boldsymbol{z_t}}. \tag{3.11}$$

Since the results previously presented are conditional on the parameter vector $\boldsymbol{\beta}$ and the discount factor γ, we will next to discuss how to obtain the posterior distributions of $\boldsymbol{\beta}$ and γ. Since these distributions can not be obtained analytically, we will use Markov chain Monte Carlo (MCMC) methods to generate samples from these posterior distributions.

4   Bayesian Analysis

Most of the parameter updating and forecasting for the dynamic model presented in Section 2 is available in closed form given that the discounting term γ is known. Alternatively, one can assume an unknown γ and use the Bayesian paradigm to carry out inference. Following the development of Section 3, the distributions obtained for the dynamic model with covariates are all conditional on $\boldsymbol{\beta}$ and γ. Our objective is to obtain the posterior joint distribution of the model parameters given that we have observed all default counts up to time t, that is $p(\theta_1, \cdots, \theta_t | N^{(t)})$ for the dynamic model and $p(\theta_1, \cdots, \theta_t, \boldsymbol{\beta} | N^{(t)})$ for the dynamic model with covariates, both of which can be used to infer mortgage default risk behavior of a given cohort. In addition, being able to obtain one month ahead predictive distributions of the default counts, $p(N_t | N^{(t-1)})$, will be of interest to institutions that are managing the loans.

4.1 Posterior Inference

Since our goal is to obtain $p(\theta_1, \cdots, \theta_t, \boldsymbol{\beta} | N^{(t)})$ which is not available in closed form we can use a Gibbs sampler to generate samples from it. In order to do so, we need to be able to generate samples from the full conditionals of $p(\theta_1, \cdots, \theta_t | \boldsymbol{\beta}, N^{(t)})$ and $p(\boldsymbol{\beta} | \theta_1, \cdots, \theta_t, N^{(t)})$, none of which are available as known densities. Next we discuss how to generate samples from these densities.

The first full conditional, that is the conditional posterior distribution of $\boldsymbol{\beta}$ given the default rates can be obtained via

$$p(\boldsymbol{\beta} | \theta_1, \cdots, \theta_t, z_t, N^{(t)}) \propto \prod_{i=1}^{t} \frac{exp\{\theta_i e^{\boldsymbol{\beta}' \boldsymbol{z_i}}\}(\theta_i e^{\boldsymbol{\beta}' \boldsymbol{z_i}})^{N_i}}{N_i!} p(\boldsymbol{\beta}),
\tag{4.1}$$

where $p(\boldsymbol{\beta})$ is the prior for $\boldsymbol{\beta}$. Regardless of the prior selection for $\boldsymbol{\beta}$, (4.1) will not be a known density. Therefore, we use a Markov chain Monte Carlo algorithm such as the Metropolis Hastings to be able to generate samples from $p(\boldsymbol{\beta} | \theta_1, \cdots, \theta_t, z_t, N^{(t)})$. Following Chib and Greenberg (1995), the steps in the Metropolis-Hastings algorithm can be summarized as follows

1. Assume the starting points $\boldsymbol{\beta^{(0)}}$ at $j = 0$.

   Repeat for $j > 0$,

2. Generate $\boldsymbol{\beta^*}$ from $q(\boldsymbol{\beta^*} | \boldsymbol{\beta^{(j)}})$ and u from $U(0, 1)$.

3. If $u \leq a(\boldsymbol{\beta^{(j)}}, \boldsymbol{\beta^*})$ then set $\boldsymbol{\beta^{(j)}} = \boldsymbol{\beta^*}$; else set $\boldsymbol{\beta^{(j)}} = \boldsymbol{\beta^{(j)}}$ and $j = j + 1$,

where

$$a(\boldsymbol{\beta^{(j)}}, \boldsymbol{\beta^*}) = min\left\{1, \frac{\pi(\boldsymbol{\beta^*})q(\boldsymbol{\beta^{(j)}} | \boldsymbol{\beta^*})}{\pi(\boldsymbol{\beta^{(j)}})q(\boldsymbol{\beta^*} | \boldsymbol{\beta^{(j)}})}\right\}.
\tag{4.2}$$

In (4.2), $q(.|.)$ is the multivariate normal proposal density and $\pi(.)$ is given by (4.1) that is the density we need to generate samples from. If we repeat the above a large number of times then we obtain samples from $p(\boldsymbol{\beta} | \theta_1, \cdots, \theta_t, z_t, N^{(t)})$. Next we discuss how one can generate samples from the other full conditional, $p(\theta_1, \cdots, \theta_t | \boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t)})$.

Due to the Markovian nature of the default rates, using the chain rule we can rewrite the full conditional density, $p(\theta_1, \cdots, \theta_t | \boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t)})$ as

$$p(\theta_t | \boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t)}) p(\theta_{t-1} | \theta_t, \boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t-1)}) \cdots p(\theta_1 | \theta_2, \boldsymbol{\beta}, \boldsymbol{z_t}, N^{(1)}).
\tag{4.3}$$

We note that $p(\theta_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t)})$ is available from (3.7) and $p(\theta_{n-1}|\theta_n, \boldsymbol{\beta}, \boldsymbol{z_t}, N^{(n-1)})$ for any n can be obtained as follows

$$p(\theta_{n-1}|\theta_n, \boldsymbol{\beta}, \boldsymbol{z_t}, N^{(n-1)}) \propto p(\theta_n|\theta_{n-1}, \boldsymbol{\beta}, \boldsymbol{z_t}, N^{(n-1)})p(\theta_{n-1}|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(n-1)}). \qquad (4.4)$$

It can be shown that $(\theta_{n-1}|\theta_n, \boldsymbol{\beta}, \boldsymbol{z_t}, N^{(n-1)}) \sim Gamma[(1-\gamma)a_{n-1}, b_{n-1}]$ where $\gamma\theta_n < \theta_{n-1} < \infty$, that is, a shifted gamma density.

Therefore, given (4.3) and the posterior samples generated from the full conditional of $\boldsymbol{\beta}$, we can sample from $p(\theta_1, \cdots, \theta_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t)})$ by sequentially simulating the individual default rates as follows

1. Assume the starting points $\theta_1^{(0)}, \cdots, \theta_t^{(0)}$ at $j = 0$.
   Repeat for $j > 0$,

2. Using the generated $\boldsymbol{\beta}^{(j)}$, sample $\theta_t^{(j)}$ from $(\theta_t|\boldsymbol{\beta}^{(j)}, \boldsymbol{z_t}, N^{(t)})$.

3. Using the generated $\boldsymbol{\beta}^{(j)}$, for each $n = t-1, \cdots, 1$ generate $\theta_n^{(j)}$ from $(\theta_n|\theta_{n+1}^{(j)}, \boldsymbol{\beta}, \boldsymbol{z_t}, N^{(n)})$ where $\theta_{n+1}^{(j)}$ is the value generated in the previous step.

If we repeat the above large number of times, then we obtain samples from the joint full conditional of default rates. This approach is also known as the *forward filtering backward sampling algorithm* [see Fruhwirth-Schnatter (1994)]. Consequently, we can obtain samples from the joint density of the model parameters via iteratively sampling from the full conditionals, $p(\boldsymbol{\beta}|\theta_1, \cdots, \theta_t, z_t, N^{(t)})$ and $p(\theta_1, \cdots, \theta_t|\boldsymbol{\beta}, \boldsymbol{z_t}, N^{(t)})$, namely a full Gibbs sampler algorithm [see for example, Smith and Gelfand (1992)].

The forward filtering backward sampling algorithm as discussed above can also be used to generate samples form $p(\theta_1, \cdots, \theta_t|N^{(t)})$ for the dynamic model without the use of the additional Gibbs sampler step for $\boldsymbol{\beta}$. In addition, the above algorithm allows us to obtain a density estimate for $p(\theta_{t-k}|N^{(t)})$ for all $k \geq 1$ for both dynamic models which can be used for retrospective comparison of default rates among different mortgage pools. To the best of our knowledge this type of approach has not been considered in the mortgage default risk literature.

4.2 Unknown Discount Parameter γ

Previously the discount factor γ has been assumed to be known. If γ were to be treated as an unknown quantity, then it is possible to carry out its Bayesian updating. Following the development of the dynamic model introduced in Section 2, the posterior distribution of γ can then be obtained via

$$p(\gamma|N^{(t)}) \propto \prod_{k=1}^{t} p(N_k|N^{(k-1)}, \gamma)p(\gamma), \tag{4.5}$$

where $p(N_k|N^{(k-1)}, \gamma)$ is the likelihood term which is given by (2.8) and $p(\gamma)$ is any choice of prior for γ. Since (4.5) will not be a known density for any choice of a prior for γ, we need to sample from the posterior distribution of γ using MCMC. As an alternative, a discrete prior over $(0, 1)$ can be considered which can numerically be summed out from (4.5).

For the dynamic model with covariates detailed in Section 3, one can generate samples from the posterior joint distribution of the discount term, γ and the covariate parameters, $\boldsymbol{\beta}$ from the following

$$p(\gamma, \boldsymbol{\beta}|N^{(t)}, \boldsymbol{z_t}) \propto p(N_1, \cdots, N_t|\boldsymbol{z_t}, \gamma, \boldsymbol{\beta})p(\gamma, \boldsymbol{\beta}), \tag{4.6}$$

where $p(\gamma, \boldsymbol{\beta}) = p(\gamma)p(\boldsymbol{\beta})$ when γ and $\boldsymbol{\beta}$ are assumed to be independent a priori and the likelihood term, $p(N_1, \cdots, N_t|\boldsymbol{z_t}, \gamma, \boldsymbol{\beta})$ can be obtained as

$$p(N_1, \cdots, N_t|\boldsymbol{z_t}, \gamma, \boldsymbol{\beta}) = L(\gamma, \boldsymbol{\beta}; \boldsymbol{z_t}, N^{(t)}) = \prod_{k=1}^{t} p(N_k|N^{(k-1)}, \boldsymbol{z_t}, \boldsymbol{\beta}, \gamma), \tag{4.7}$$

where $p(N_k|N^{(k-1)}, \boldsymbol{z_t}, \boldsymbol{\beta}, \gamma)$ is given by (3.10). We note here that (4.7) is free of θ's which facilitates the posterior generation. Since (4.6) will not be available in closed form for any prior choice of γ and $\boldsymbol{\beta}$, one can use a Metropolis-Hastings algorithm to generate samples from the joint posterior density as presented in Section 3.1.

As a result, the conditional joint distribution of the default rates, $p(\theta_1, \cdots, \theta_t|N^{(t)}, \boldsymbol{z_t}, \boldsymbol{\beta}, \gamma)$ using the forward filtering backward sampling algorithm as presented in Section 3.1. Thus, the joint smoothing distribution of the default rates can be computed via

$$p(\theta_1, \cdots, \theta_t|N^{(t)}) = \int \int p(\theta_1, \cdots, \theta_t|N^{(t)}, \boldsymbol{z_t}, \boldsymbol{\beta}, \gamma)p(\gamma, \boldsymbol{\beta}|N^{(t)})d\gamma d\boldsymbol{\beta}, \tag{4.8}$$

where only samples from $p(\gamma, \boldsymbol{\beta}|N^{(t)})$ will be available. Therefore the above can be approximated as a Monte Carlo average via

$$p(\theta_1, \cdots, \theta_t|N^{(t)}) \approx \frac{1}{S}\sum_{j=1}^{S} p(\theta_1, \cdots, \theta_t|N^{(t)}, \boldsymbol{z_t}, \boldsymbol{\beta^{(j)}}, \gamma^{(j)}), \qquad (4.9)$$

where S is the number of samples, $(\boldsymbol{\beta^{(j)}}, \gamma^{(j)})$ are the generated sample pairs.

4.3   One Month Ahead Forecasting

In order to obtain one month ahead forecast distributions from the dynamic model with covariates, the following can be used

$$p(N_t|N^{(t-1)}, \boldsymbol{z_t}) = \int\int p(N_t|N^{(t-1)}, \boldsymbol{z_t}, \boldsymbol{\beta}, \gamma)p(\gamma, \boldsymbol{\beta}|N^{(t)})d\boldsymbol{\beta}d\gamma. \qquad (4.10)$$

Since only samples from $p(\gamma, \boldsymbol{\beta}|N^{(t)})$ will be available, the above can be approximated via

$$p(N_t|N^{(t-1)}, \boldsymbol{z_t}) \approx \frac{1}{S}\sum_{j=1}^{S} p(N_t|N^{(t-1)}, \boldsymbol{z_t}, \boldsymbol{\beta^{(j)}}, \gamma^{(j)}). \qquad (4.11)$$

Similarly, (4.11) can be computed for the dynamic model of Section 2 without any covariates.

4.4   Model Comparison

In order to assess and compare the fit performance for the proposed models, we consider two sets of measures that are used with sampling based methods, the Bayes factor with the harmonic mean estimator and the pseudo Bayes factor with the conditional predictive ordinate. In what follows, we briefly summarize both methods whose implementations are discussed in our numerical example.

4.4.1   Bayes Factor-Harmonic Mean Estimator

The first fit measure is the Bayes factor approximation of models with MCMC steps, we refer to this measure as the Bayes Factor-Harmonic Mean Estimator which has been discussed by Gelfand et al. (1992) and Kass and Raftery (1995). The harmonic mean estimator of the predictive likelihood for

a given model can be obtained as

$$p(\mathbf{D}) = \{\frac{1}{S} \sum_{j=1}^{S} p(\mathbf{D}|\mathbf{\Theta}^{(j)})^{-1}\}^{-1}, \tag{4.12}$$

where S is the number of iterations and $\mathbf{\Theta}^{(j)}$ is jth generated posterior sample. For the proposed models, (4.12) can be computed via

$$p(\mathbf{D}) = \{\frac{1}{S} \sum_{j=1}^{S} \{\prod_{k=1}^{t} p(N_k|N^{(k-1)}, \mathbf{\Theta}^{(j)})\}^{-1}\}^{-1}, \tag{4.13}$$

where $p(N_k|N^{(k-1)}, \mathbf{\Theta}^{(j)}) = p(N_k|N^{(k-1)}, \gamma^{(j)})$ can be obtained via (2.8) and $p(N_k|N^{(k-1)}, \mathbf{\Theta}^{(j)}) = p(N_k|N^{(k-1)}, \mathbf{z_t}, \boldsymbol{\beta}^{(j)}, \gamma^{(j)})$ via (3.10) for the dynamic models without and with covariates, respectively. In comparing two models, a higher $p(\mathbf{D})$ value indicates a better fit. As pointed out by Kass and Raftery (1995), although the use of (4.12) has been criticized due to potential large effects of a sample value on the likelihood, it has been shown to give accurate results in most cases and is preferred for its computational simplicity.

4.4.2 Pseudo Bayes Factor-Conditional Predictive Ordinate

An alternative method to compare models with sampling based estimation is the calculation of the pseudo Bayes factor using the conditional predictive ordinate. Following Gelfand (1996), the comparison criteria makes use of a cross-validation estimate of the marginal likelihood. The main advantage of this approach is once again its computational simplicity.

The cross validation predictive density for the ith observation is defined as $f(N_i|\mathbf{N}^{(-i)})$, where $\mathbf{N}^{(-i)}$ represents the data, $N^{(i)}$, except for N_i and can be estimated via

$$\hat{f}(N_i|\mathbf{N}^{(-i)}) = \frac{1}{\frac{1}{S} \sum_{j=1}^{S} \frac{1}{f(N_i|\mathbf{N}^{(-i)}, \mathbf{\Theta}^{(j)})}}, \tag{4.14}$$

where S is the number of samples generated and $\mathbf{\Theta}^{(j)}$ is the jth generated parameter sample vector. Since given $\mathbf{\Theta}$, N_is are independent, $f(N_i|\mathbf{N}^{(-i)}, \mathbf{\Theta}^{(j)}) = f(N_i|\mathbf{\Theta}^{(j)})$ can be used in (4.14). Once the cross validation predictive densities are estimated using (4.14), one can compare the proposed models in terms of fit in the log-scale. In comparing models, a higher conditional predictive ordinate

indicates a better fit.

4.4.3 A Benchmark Model; a Bayesian Poisson Regression

As pointed out earlier, a Bayesian Poisson regression model can be used to test the dynamic nature of the default rate and also can act as a benchmark model for an out of sample forecasting exercise. In this case, we assume that the default counts, N_ts follow a non-homogeneous Poisson process whose default rate is θ_t where $\theta_t = exp\{\boldsymbol{\beta'z_t}\}$, in other words the default rate is a deterministic function of the covariates and is not stochastically evolving over time as compared to the previously proposed dynamic models. In order to obtain the posterior distribution of the model parameters, $\boldsymbol{\beta}$, we can use the Metropolis-Hastings algorithm as discussed in Section 3.1. where the likelihood function is given by

$$L(\boldsymbol{\beta}; N^{(t)}, \boldsymbol{z_t}) = \prod_{i=1}^{t} \frac{exp\{e^{\boldsymbol{\beta'z_i}}\}(e^{\boldsymbol{\beta'z_i}})^{N_i}}{N_i!}, \tag{4.15}$$

and each β coefficient is a priori, assumed to be normally distributed.

5 Analysis of Monthly Mortgage Default Counts

In order to illustrate how the proposed models can be applied to real mortgage default risk, we have used the data provided by Federal Housing Administration (FHA) of the U.S. Department of Housing and Urban Development (HUD). The data consists of defaulted FHA insured single family mortgage loans originated in different years and in four regions where HUD has local offices. In our analysis of the default counts, we use a subset of the data which consists of defaulted FHA insured single-family 30-year fixed rate (30-yr FRM) mortgage loans from 1994 in the Atlanta region.

Since default behavior is influenced by factors relating to both the housing equity and the mortgage borrower's ability to pay the loan, we consider two equity and two ability-to-pay covariates in our analysis. Housing equity is mainly determined by the housing price level and interest rate. Therefore we include regional conventional mortgage home price index (CMHPI) and federal cost of funds index (COFI) as aggregate equity factors. The CMHPI and COFI are provided by Freddie Mac, and are used as benchmark indices in the U.S. residential mortgage market. In addition, in order to take into account borrowers' overall repayment ability, we consider the homeowner mortgage financial obligations ratio (FOR Mortgage) from The Federal Reserve Board which reflects

periodical mortgage repayment burden of borrowers, and regional unemployment rate from the U.S. Census, which represents the impact from trigger events at the aggregate level.

As seen in Figure 1, the default counts for the 1994 cohort seem to exhibit a non-stationary behavior which can be captured by our state space models. In what follows we illustrate the implementation of each model to the data and discuss implications and relevant fit measures.
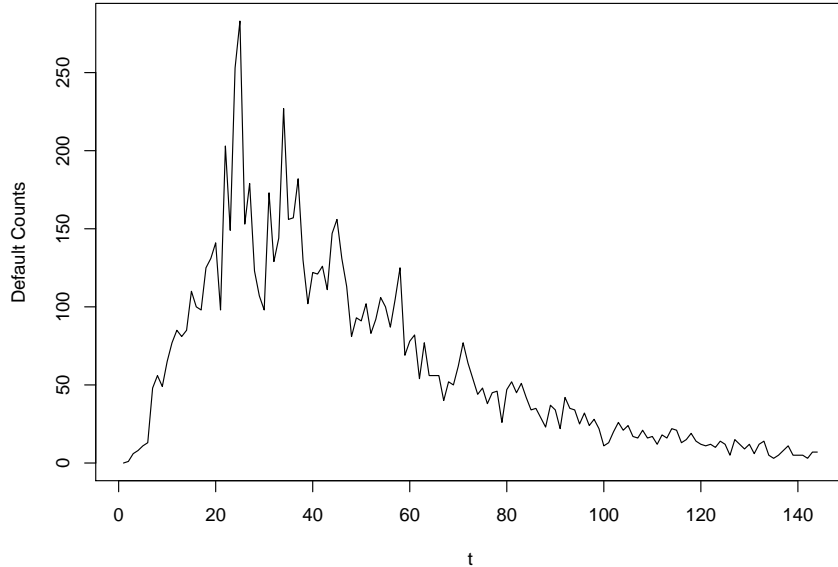

Figure 1: Monthly default counts for the 1994 Cohort

5.1 Dynamic Model without Covariates

As discussed in Section 2, the dynamic model assumes that the default counts are observations from a non-homogeneous Poisson process whose rate is stochastically evolving over time. The attractive feature of the dynamic model with no covariates is its analytical tractability and straight forward updating scheme. In our analysis, we have assumed that the discounting factor γ given in (2.2) follows a discrete uniform distribution defined over $(0, 1)$ and have obtained its posterior density via (4.5). As shown in Figure 2, the posterior distribution of γ is concentrated around 0.15 and 0.32 with a mean of 0.23. Using the posterior of γ and the forward filtering backward sampling algorithm presented in Section 4.1, one can obtain the retrospective fit of the default rate

given data. An overlay plot of the mean posterior default rate and the actual data is shown in Figure 2 where evidence in favor of the proposed dynamic model can be inferred. Given the joint distribution of the default rate over time, i.e. $p(\theta_1, \cdots, \theta_t | N^{(t)})$, the financial institution managing the loans will have a better understanding of the default behavior of a given cohort and can use it to manage risk or explain potential behavior of similar cohorts. In addition, Bayesian analysis of the mortgage default risk allows direct comparison of the default rates during different time periods probabilistically. For instance, one can compute the posterior probability that default rate during the second month is greater than that of the first month for a given cohort, that is $p(\theta_2 \geq \theta_1 | N^{(t)})$ which can be computed to be 0.3387.
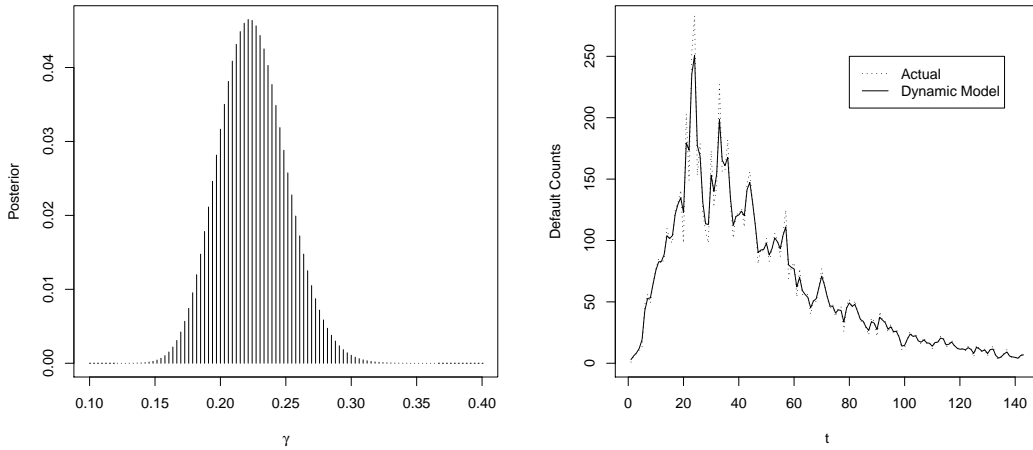

Figure 2: Posterior γ of the dynamic model (left) and the retrospective fit of the dynamic model to data (right)

5.2 Dynamic Model with Covariates

In taking into account the effects of macroeconomic variables, we have implemented the dynamic model with covariates as presented in Section 3. In doing so, we have assumed flat but proper priors for the model parameters. More specifically, the discounting term, γ, a priori follows a uniform distribution defined over $(0, 1)$ and the covariate coefficients, $\boldsymbol{\beta}$, independent normal distributions. We ran the Markov chain Monte Carlo algorithm for 10,000 iterations with a burn-in period of 2,000 iterations, and have not encountered any convergence issues. The trace plots for the posterior samples are shown in Table 3 based on which it can be concluded that convergence has been
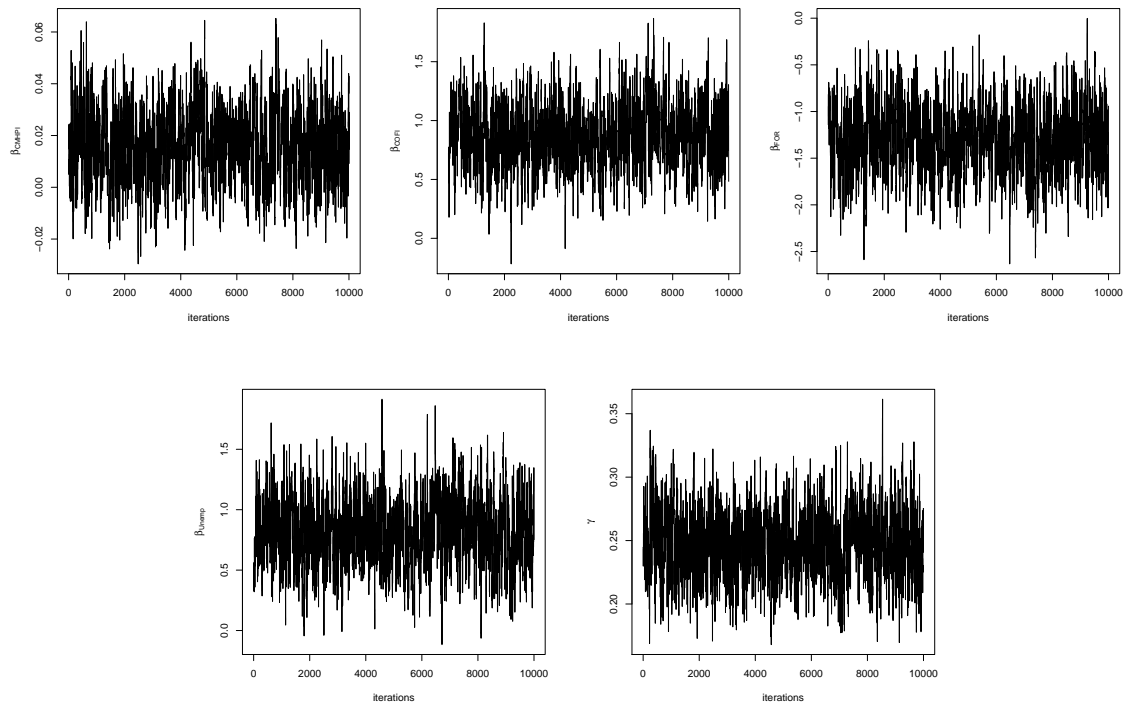
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attained.

Figure 3: Trace plots of $\boldsymbol{\beta}$ and γ of the dynamic model with covariates

The posterior density plots of $\boldsymbol{\beta}$ are shown in Figure 4 and of γ in Figure 5 which exhibits similar behavior to the posterior discounting term obtained for the dynamic model as in Figure 2.
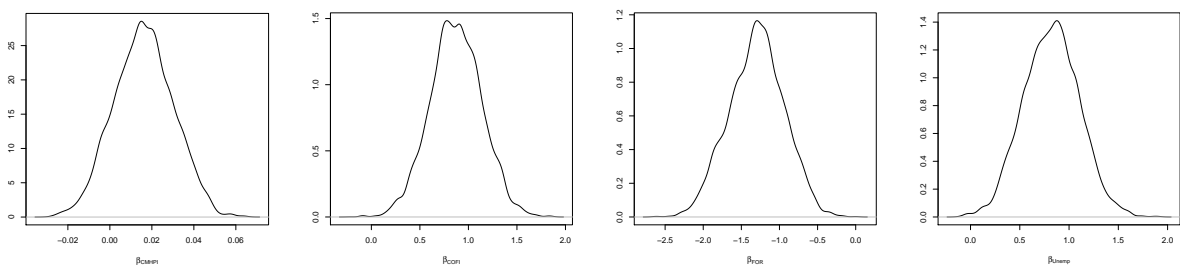

Figure 4: Posterior density plots of $\boldsymbol{\beta}$ of the dynamic model with covariates

As can be observed from Table 1, the β coefficients all seem to have fairly significant effects on the default rate. An advantage of the Bayesian approach is its ability to quantify posterior inference probabilistically. For instance, one can calculate the probability that β_{CMHPI} is greater
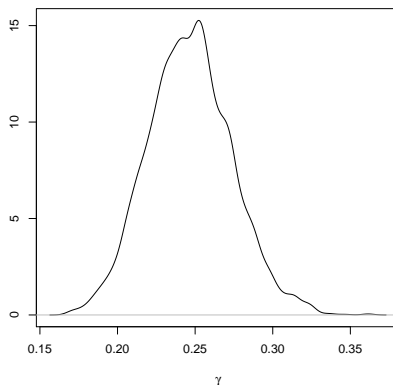
Figure 5: Posterior density plot of γ of the dynamic model with covariates

than 0, i.e. $p(\beta_{CMHPI} > 0|N^{(t)})$. Given the cohort at hand, $P(\beta_{CMHPI} > 0|N^{(t)})$ was obtained to be approximately 0.87 which shows strong evidence in favor of a positive effect. In summary, the regional conventional mortgage home price index (CMHPI), federal cost of funds index (COFI) and the regional unemployment rate (Unemp) have positive effects on default counts. For instance, as unemployment tends to go up, the model suggests that the number of people defaulting tend to increase for the cohort under study. On the other hand, the homeowner financial obligations ratio (FOR) seem to decrease the expected number of defaults as it goes up, namely as the burden of repayment becomes relatively easier then home owners are less likely to default.

Statistics	β_{CMHPI}	β_{COFI}	β_{FOR}	β_{Unemp}	γ
25th	0.0063	0.7003	-1.5430	0.6252	0.2281
Mean	0.0160	0.8717	-1.3002	0.8191	0.2466
75th	0.0256	1.0510	-1.0550	1.0117	0.2643
St.Dev	0.0141	0.2663	0.3606	0.2826	0.0270

Table 1: Posterior statistics for $\boldsymbol{\beta}$ and γ of the dynamic model with covariates

One of the issues that had been under investigation so far was the dynamic nature of the default rate. As shown in the right panel of Figure 7, the fit of dynamic model with covariates is reasonably good, justifying the dynamic behavior of default rates. A similar conclusions can be drawn for the dynamic model without the covariates whose fit is shown in the right panel of Figure 2. In showing the dynamic nature of the default rate, we have obtained the joint distribution of the baseline default rates, that is $p(\theta_1, \cdots, \theta_t|N^{(t)})$ as in (4.9). A boxplot of θ_ts is shown in Figure 6, which

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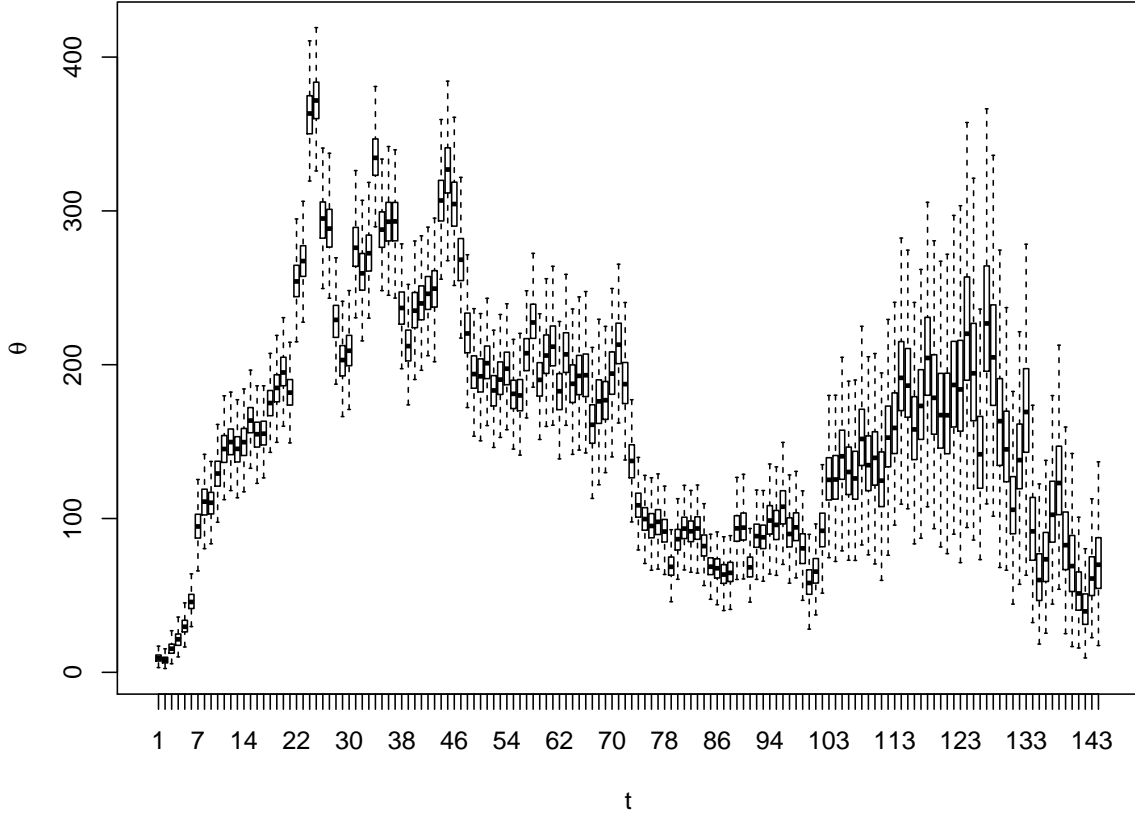provides strong evidence in favor of a dynamic default rate.

Figure 6: Boxplots for smoothed θ's from $p(\theta_1, \cdots, \theta_t | N^{(t)})$

5.3 Comparison with the Bayesian Poisson Regression Model

For the estimation of the Bayesian Poisson regression model, we have assumed, a priori, that the β coefficients are independently normally distributed and implemented the MCMC algorithm as discussed in Section 4.4. The posterior statistics of the β coefficients are shown in Table 2 from which conclusions similar to that of the previous model can be drawn. Namely, all macroeconomic variables seem to have an impact on the default counts and except for the regional conventional mortgage home price index (CMHPI), the coefficient signs seem to be consistent with those of the dynamic model with covariates. As for the CMHPI, $p(\beta_{CMHPI} > 0 | N^{(t)})$ was obtained to be

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approximately 0.28 which maybe due to the dynamic nature of the baseline default rate, θ_t. In other words, if the dynamic default rate is not taken into account, then the effects of the CMHPI might have been suppressed in the Poisson regression model.

Statistics	β_0	β_{CMHPI}	β_{COFI}	β_{FOR}	β_{Unemp}
25th	1.8614	-0.0023	0.6827	-0.4503	0.3146
Mean	2.2344	-0.0011	0.7114	-0.3710	0.3583
75th	2.6040	0.0022	0.7392	-2.9400	0.4013
St.Dev	0.5431	0.0017	0.0411	0.1122	0.0621

Table 2: Posterior statistics for $\boldsymbol{\beta}$ of the Bayesian Poisson regression model

Furthermore, the retrospective fit for the Bayesian Poisson regression model as shown in the left panel of Figure 7 suggests a lack of fit. In other words, there is evidence in favor of a dynamic default rate which was captured by the two proposed dynamic models.
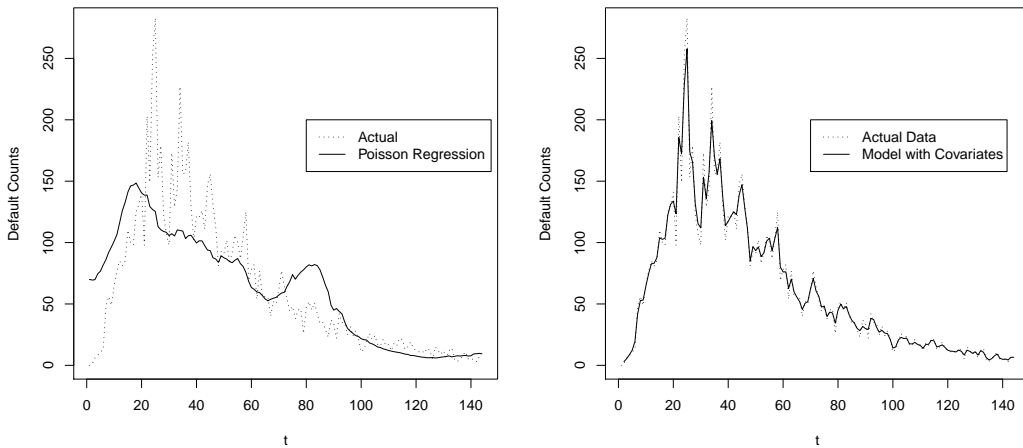

Figure 7: Retrospective fit of the Poisson regression model and dynamic model with covariates to data

5.4 Model Comparison

In order to compare the fit of the proposed models, we computed the log-marginal likelihoods as given by (4.13) and the conditional predictive ordinates in the log-scale as given by (4.14). The results are shown in Table 3 where DM1 stands for the dynamic model, DM2 for the dynamic model with covariates and BPM for Bayesian Poisson regression model. The dynamic model with

covariates (DM2) has the highest log-marginal likelihood value and the highest CPO with a Bayes factor of approximately 10.3 (BF=$\frac{p(\boldsymbol{D}|\text{DM1})}{p(\boldsymbol{D}|\text{DM2})}$) which according to Kass and Raftery (1995), shows strong support in favor of DM2. The results further support the lack of fit of the static model and shows decisive evidence in favor of either one of the dynamic models with a Bayes factor of > 100.

	DM1	DM2	BPM
$\log\{p(\boldsymbol{D})\}$	-579.99	-577.61	-1416.28
$\log(\text{CPO})$	-580.07	-578.69	-1372.62

Table 3: $\log\{p(\boldsymbol{D})\}$ and $\log(\text{CPO})$ under each model

In addition to understanding the default behavior of a given cohort, it is also of interest to assess the model's ability to predict future defaults given the past. To assess the forecasting performance of the proposed models, we have considered the first 134 months of data as the training set and have sequentially predicted 10 future months. To provide one month ahead forecasting comparisons, we have considered two measures; the mean absolute percentage error (MAPE) and the root mean squared error (RMSE) calculated as

$$\text{MAPE} = \frac{1}{10} \sum_{i=1}^{10} \frac{|N_i - E(N_i)|}{N_i}, \tag{5.1}$$

where N_i is the actual default count observed during the ith month, $E(N_i)$ is its one month-ahead prediction. Similarly,

$$\text{RMSE} = \sqrt{\frac{1}{10} \sum_{i=1}^{10} \{N_i - E(N_i)\}^2}. \tag{5.2}$$

The forecasting performance results are shown in Table 4 where the dynamic models seem to exhibit better forecasting performance than the static model. An interesting finding is that the dynamic model seem to provide the best set of forecasts for this particular 10 data points even though the overall model fit of the dynamic model with covariates had been concluded to be superior.

	DM1	DM2	BPM
MAPE	10.5	50.6	103.1
RMSE	0.63	3.02	4.93

Table 4: Forecasting performance comparison

5.5 Analysis of 1995 and 1996 Cohorts

In order to investigate whether additional insights can be gained by using a different mortgage cohort, we applied the proposed models to cohorts which consist of defaulted FHA insured single-family 30-year fixed rate (30-yr FRM) mortgage loans from 1995 and 1996. In terms of fit, similar results have been obtained, that is, strong evidence in favor of dynamic default behavior has been established. On the other hand, as shown in Tables 5 and 6 the distributions of the covariate coefficients slightly differ from that of the 1994 cohort. For instance, the posterior mean of β_{CMHPI} is now estimated to be negative for both 1995 and 1996 cohorts. This simply implies that the default behavior for different cohorts differ from each other, which in turn validates the need for proper modeling of the default risk at the aggregate level.

Statistics	β_{CMHPI}	β_{COFI}	β_{FOR}	β_{Unemp}	γ
25th	-0.0125	-0.0955	-0.4960	0.2454	0.2560
Mean	-0.0040	0.0818	-1.2690	0.4269	0.2771
75th	0.0042	0.2650	-0.0468	0.6067	0.2978
St.Dev	0.0127	0.2673	0.3336	0.2663	0.0304

Table 5: Posterior statistics for $\boldsymbol{\beta}$ and γ of the dynamic model with covariates for the 1995 cohort

Statistics	β_{CMHPI}	β_{COFI}	β_{FOR}	β_{Unemp}	γ
25th	-0.0113	0.0529	-0.4438	0.0746	0.1833
Mean	-0.0033	0.2138	-0.2229	0.2509	0.2014
75th	0.0044	0.3717	0.0023	0.4292	0.2182
St.Dev	0.0119	0.2394	0.3239	0.2588	0.0251

Table 6: Posterior statistics for $\boldsymbol{\beta}$ and γ of the dynamic model with covariates for the 1996 cohort

6 Concluding Remarks

In this paper we considered discrete time Bayesian state space models with Poisson measurements to model the aggregate mortgage default risk. As pointed out by Kiefer (2011) the Bayesian approach provides a coherent framework to combine data with prior information and enables us make inferences using probabilistic reasoning. In addition, proposed discrete time Bayesian state space models with stochastic default rate can capture the effects of correlated defaults over time. In order to carry out inference on model parameters, we have made use of Markov chain Monte

Carlo methods such as the Gibbs sampler, Metropolis-Hastings and forward filtering backward sampling algorithms. In assessing the dynamic nature of the mortgage default rate, we compared the forecasting performance of the proposed models with a Bayesian Poisson regression model used as a benchmark. We illustrated the use of the proposed models using actual U.S. residential mortgage data and discussed insights gained from Bayesian analysis. To the best of our knowledge these type of models from a Bayesian point of view has not been considered in the default risk literature at the aggregate level and can be considered to be novel contributions of our proposed approach.

In modeling the aggregate mortgage default risk we addressed whether the default rate was exhibiting static or dynamic behavior and investigated the effects of macroeconomic variables on default risk. Strong evidence in favor of dynamic default behavior at the aggregate level has been found. Furthermore, we also found significance effects of macroeconomic variables such as the regional conventional mortgage home price index, federal cost of funds index, the homeowner mortgage financial obligations ratio and the regional unemployment rate on the aggregate mortgage default risk.

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