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REVERSE-ENGINEERING COUNTRY RISK RATINGS: A COMBINATORIAL NON-RECURSIVE MODEL

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Abstract: The central objective of this paper is to develop a transparent, consistent, self-contained, and stable country risk rating model, closely approximating the country risk ratings provided by Standard and Poor's (S&P). The model should be non-recursive, i.e., it should not rely on the previous years' S&P ratings. The set of variables selected here includes not only economic-financial but also political variables. We propose a new model based on the novel combinatorial-logical technique of Logical Analysis of Data (which derives a new rating system only from the qualitative information representing pairwise comparisons of country riskiness). We also develop a method allowing to derive a rating system that has any desired level of granularity. The accuracy of the proposed model's predictions, measured by its correlation coefficients with the S&P ratings, and confirmed by *k*-folding cross-validation, exceeds 95%. The stability of the constructed non-recursive model is shown in three ways: by the correlation of the predictions with those of other agencies (Moody's and The Institutional Investor), by predicting 1999 ratings using the non-recursive model derived from the 1998 dataset applied to the 1999 data, and by successfully predicting the ratings of several previously non-rated countries. This study provides new insights on the importance of variables by supporting the necessity of including in the analysis, in addition to economic variables, also political variables (in particular "political stability"), and by identifying "financial depth and efficiency" as a new critical factor in assessing country risk.

Key words: combinatorial optimization, logical analysis of data, rating, credit risk, country risk

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1 Country Risk, Country Risk Ratings and Objectives of the Paper

1.1 Country risk, country risk ratings and their importance

The globalization of the world economies, and in particular the internationalization of financial markets in the last decades, have dramatically expanded and diversified investment possibilities, leading to numerous new opportunities, accompanied by new risks. Consequently, there has been growing interest in obtaining reliable estimates of the risk of investing in different countries. The importance of ratings has been magnified by the recommendations addressed in the Basel Capital Accord (2005), that pinpoints the role of agencies' ratings for the assessment of credit risk.

In response to the increased demand for the evaluation of creditworthiness, rating agencies such as Moody's, Standard & Poor's (S&P), Fitch Ratings, The Institutional Investor, etc. are providing ratings, or scores, indicative of possible future default. Country credit risk ratings are defined by Eliasson (2002) as the "risk of national governments defaulting on their obligations", while Afonso et al. (2007) state that "sovereign credit ratings are a condensed assessment of a government's ability and willingness to repay its public debt both in principal and in interests on time".

Country (or sovereign) risk ratings impact countries in a number of ways. The primary significance of ratings is due to their influence on the interest rates at which countries can obtain credit on the international financial markets: the higher the ratings (i.e., the lower the risk of default) the lower the interest rate. Following its sovereign rating downgrade, Japan's borrowing became more expensive as interest rates have increased, reflecting the higher chance of default, which deteriorates even more the situation of the heavily indebted Japanese economy.

Second, country risk ratings also influence credit ratings of national banks and companies, and affect their attractiveness to foreign investors. Ferri et al. (1999) call sovereign ratings the "pivot of all other country's ratings", while Eliasson (2002), Durbin and Ng (2005) and Mora (2006) consider the country risk rating as the credit risk ceiling for all obligors located in a country. Indeed, Similarly, raters have historically been reluctant to give a company a higher credit rating than that of the sovereign where the company operates (Erb et al., 1996). For example, after Moody's downgraded Japan in November 1998 (from Aaa to Aa1), all other Aaa Japan issuers have been downgraded (Jüttner, McCarthy, 2000). This led credit risk ratings to be named "credit risk ceilings".

Third, institutional investors are sometimes contractually restricted on the degree of risk they can assume, implying in particular that they cannot invest in debt rated below a prescribed level. Ferri et al. (1999) refine this analysis, pointing out the contrast between the ratings of banks operating in high- and low-income countries, and show that ratings of banks operating in low-income countries are significantly affected by variations in sovereign ratings, while the ratings of banks operating in high-income countries (Kaminsky, Schmukler, 2002, Larrain et al., 1997) do not depend that much on country ratings.

The existing literature on country risk, i.e. the risk that a country defaults on its obligations, recognizes both financial/economic and political components of country risk. There are two basic approaches to the interpretation of the reasons for defaulting. The debt-service capacity approach focuses on the deterioration of solvency of a country, which prevents it from fulfilling its commitments. For instance, Bourke and Shanmugam (1990) define country risk as "the risk that a country will be unable to service its external debt due to an inability to generate sufficient foreign exchange". Within this framework, country risk is viewed as a function of various financial and economic country parameters. The cost-benefit approach views a default on commitments or a rescheduling of debt as a deliberate choice of the country, which may prefer this alternative over repayment, in spite of its possible long-term negative effects (e.g. the country's exclusion from certain capital markets (Reinhart, 2002, reputation damage). Since the deliberate decision to default results from a political process, political country parameters are included in this type of country risk modeling, along with the financial and economic ones (Brewer, Rivoli, 1990, 1997, Citron, Neckelburg, 1987).

1.2 Critiques of present rating systems

The purpose of ratings is that of compressing a variety of information about a country into a single parameter which can be easily understood, and therefore conveniently used in a decision making process involving comparisons between different countries. Consequently, ratings provide aggregations of diverse indicators into a single metric and can be viewed as a kind of "commensuration" (Kunczik, 2000). The interpretation of ratings is complicated by the heterogeneity of indicators (political stability, inflation, etc.) which may have been used in deriving them.

Comprehensibility: Rating agencies do not specify the factors that are used for determining their ratings, nor the way they are aggregated in a rating. It is generally assumed that ratings are obtained by aggregating economic/financial and/or political variables. Clearly, the main objective of any country risk rating system is to represent the creditworthiness of countries. It is not clear however which ones of the many possible factors do actually influence the payback capacity of a country. This question is subject to different analyses. Haque et al. (1998) claim that it is sufficient to restrict the scope of analysis to economic factors only, while others (Brewer and Rivoli, 1990) claim that both economic and political factors impact country risk ratings. This lack of clarity raised the discontent of Japan's Prime Minister, Junichiro Koizumi, who was "railed at being rated in the same neighborhood as African countries to which Japan is providing assistance". In view of such controversy, uncovering both the factors which are taken into account by these black boxes, and the mechanisms of deriving ratings, are essential for ascertaining the consistency of a country rating system.

Regional bias: Haque et al. (1997) claim that (some) rating agencies favor certain regions. More precisely, they say that Euromoney tends to give better ratings to Asian and European countries than to Latin or Caribbean ones, and that the Institutional Investor is more favorable towards Asian and European countries than its is towards African ones.

Predictive power: Some recent failures (no warning ahead of the Tequila and the Asian crises) have challenged the trustworthiness of country risk ratings. Indeed, the tequila crisis in Mexico (1994-95) had not been preceded by a rating downgrade, implying that either the crisis was not predicted, or that its significance was overlooked. Similar observations apply to the Asian crisis (1997-99): Fitch admitted that "it and its larger rivals Standard's & Poor's and Moody's Investors Services of the US had largely failed to predict the recent turmoil in Asia". On the other hand, rating agencies have been more insightful in anticipating other crises, e.g. in Russia (1998), Brazil (1998) and Argentina (2001). Rating agencies have been criticized for the time they need (Altman, Rijken, 2004) to adjust the ratings and for being reactive rather predictive.

Overreactions: Rating agencies are considered to sometimes react in panic after realizing they fail to warn about a crisis, leading to the so-called *procyclicality* effect (Ferri et al., 1999). Having first failed in predicting the Asian crisis, rating agencies then reacted by severely downgrading countries such as Thailand or South Korea, thus accelerating the flight of capital. In this and other situations, rating agencies gave the impression of overreacting instead of being a stabilizing force. It appears that the objectivity and reliability of country risk ratings is questionable, mainly because of human intervention and conflicting goals and/or interests.

Negative impact of rating changes: It is reported that the reluctance of raters to downgrade a country stems from the fact that a downgrade announcement can precipitate a country into crisis. During the Asian crisis, the rating agencies arouse the discontent of the Malaysian Prime Minister who charged them with rendering the crisis even more acute: "The rating agencies, when we have a need to borrow money, they immediately downgraded us so that it will cost us 15% to borrow money. They stop us completely from borrowing money" (1999)[1]. It is argued that rating agencies lagging behind rather than anticipating the state of financial markets reinforce positive expectations and capital inflows when they upgrade countries and intensify outflows of capital and crisis when they downgrade.

Conflicts of interest: An even more pointed criticism is that raters, charging fees to rated countries, can be suspected of reluctance to downgrade them, because of the possibility of jeopardizing their income sources. This is claimed, for example, by Tom McGuire, an executive vice-president of Moody's, who states that "the pressure from fee-paying issuers for higher ratings must always be in a delicate balance with the agencies' need to retain credibility among investors"[2]. The necessity to please the payers of the ratings, investors as well as issuers, lead to what Robert Grossman, the chief credit officer at the rating agency Fitch, calls "a tendency we do with investors – rating committees, outlooks, meetings, then the press release, all to soften the blow of the rating change"[3]. Studying the rating transitions, Altman and Saunders (1998) notice that a downgrade in the rating of a country is regularly followed by further downward adjustments. The explanation given by Altman and Saunders is that agencies gradually

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[1] The article `We had to decide things for ourselves' appeared in the February 19, 1999 issue of the *Executive Intelligence Review*. Interview by G.G. Billington and D. de Paoli of Datuk Seri Dr. Mahathir bin Mohamad, Malaysian Prime Minister.

[2] The Economist, July 15, 1995, 62

[3] Euromoney, January 2002, 38, "*Investors turn cool on the rating game*"

downgrade the rating of a country, since they do not want to hurt the country, which is also their client. Kunczik (2001) note that the IMF (1999) fears the danger that “issuers and intermediaries could be encouraged to engage in *rating shopping* – a process in which the issuer searches for the least expensive and/or least demanding rating”. The issue of determining how rating agencies should be paid and how can conflicts of interests be managed is addressed in the special issue of the Journal of Banking and Finance devoted to the recent development in credit ratings (Cantor, 2004).

The problems described above will become more acute as the role of ratings increases. Indeed, the Basel Accord will intensify the pressure on countries to obtain high ratings, potentially leading to a switch from rating shopping to rating fraud. For instance, Pakistan was forced to pay back $55 million credits to the IMF because of budget falsification, the blame being put on the former Prime Minister Nawaz Sharif, accused of having falsified the budget deficit. Similarly, Ukraine has been proven to have reported misleading data on its reserves in foreign exchanges, attempting to obtain IMF credits. Kunczik (2001) says that “it is only a question of time when firms will specialize in rating advising for sovereigns”.

1.3 Recursive versus non-recursive models

The recent literature on country risk ratings contains several studies (Cantor, Packer, 1996, Haque et al., 1998, Monfort, Mulder, 2000) which use multiple regression. The set of independent variables used by Haque et al. (1998), Monfort, Mulder (2000), Afonso (2003) includes the lagged sovereign ratings of S&P, or Moody’s. The correlation levels between the ratings of various agencies and the predicted values or ratings obtained using multiple regression models referenced above are remarkably high.

To illustrate the actual meaning of these results, we shall examine the approach taken by Haque et al. (1998). That paper uses as its eight independent variables seven macro-economic variables, and the lagged rating, i.e. it includes The Institutional Investor ratings both at times t and $t-1$, the former as a dependent variable, and the latter as an independent one. We note that country risk ratings are very stable, as shown by the transition probabilities (Table 4) of the ratings published by S&P (2007).

The 98% correlation level between The Institutional Investor ratings published respectively in September 1997 and September 1998 confirms the stability property of sovereign ratings. In light of this fact, the excellent correlation levels achieved by utilizing lagged ratings among the independent variables can be attributed to a certain – possibly large – extent to this stability, and may not necessarily give indications about the predictive power of the economic and political variables used as predictors.

Although Cantor and Packer (1996) do not include the lagged ratings in their set of predictors, they create a dummy variable, which is determined by the past ratings issued by S&P (Claessens and Embrechts, 2002). This dummy variable is defined to be equal to 1 if a country has ever been rated D or SD by S&P’s since 1970, and equal to 0 otherwise. Even though their regression R-square is above 90%, their results are criticized by Jüttner and McCarthy (2000) and Claessens and Embrechts (2002). These latter mention that the dates of the explanatory variables are not consistent, e.g. the values of some variables are measured in 1994 or 1995, while that of others are averages for the period 1991-1994 or 1992-1994. On the other hand, Jüttner and McCarthy evaluated the regression model of Cantor and Packer for some other years, concluding that for 1998, it loses its predictive power. A paper of Hu et al. (2002) develops a model using ordered probit to estimate country ratings. Their model has an 83% correlation level and relies on economic variables and rating history of countries. A common feature of the econometric models above is the direct or indirect inclusion of information from past ratings (lagged ratings, rating history) among their independent variables. A major drawback of such rating models is the impossibility of applying them to not-yet-rated countries.

1.4 Objectives, main results and paper structure

Our discussion in the preceding subsections indicates a need for making country risk ratings more (i) transparent and (ii) consistent. A third criterion we would like to impose on an ideal country risk rating system is that of (iii) self-containment, i.e. its non-reliance on any other past or present country risk ratings. Clearly, this requirement precludes the use of lagged ratings as independent variables. It is important to note that this approach is in marked contrast with that of the current literature (discussed in the previous subsection), which does rely in one form or another on lagged ratings. Finally, a fourth requirement imposed on the model is its (iv) stability, i.e. extensibility both to subsequent years and to previously non-rated countries.

The wide acceptance of several of the major rating systems indicates that, while they may not be perfect, they provide the currently best known evaluation of country risk. It is therefore reasonable to base the design of any new rating system on one of the existing ones.

The **central objective** of this paper is to develop a transparent, consistent, self-contained, and stable rating system, closely approximating the learned country risk ratings. We have selected the Standard & Poor's country risk rating system as the benchmark for the desired system. It is to be expected that, on the one hand, in most cases the ratings of the new system should closely resemble those of S&P, and on the other hand, in the cases where the two ratings differ, the objective reasons, which determine the ratings of the proposed model, should be justified by subsequent developments.

We propose a new model for reverse-engineering S&P country risk ratings: it uses the novel combinatorial-logical technique of Logical Analysis of Data (LAD) which derives a new rating system only from the qualitative information representing pairwise comparisons of country riskiness. A related study (Hammer et al., 2006) also utilized LAD to reverse-engineer S&P country risk ratings. The methodology proposed in Hammer et al. (2006) and that of this paper both rely on using LAD to derive relative preferences of a country over another one. However, the two methodologies differ significantly in their subsequent steps. First, the present study develops an L_2-approximation of the LAD relative preferences, while, in Hammer et al. (2006), the creditworthiness dominance relationship of countries was directly inferred from the LAD relative preferences. Second, the methodology proposed in Hammer et al. (2006) requires the following steps for the derivation of the risk ratings: (i) the construction of the so-called base dominance relationship that itself involves the derivation of the external preference of a country over another and the expected value and standard deviation of the external preferences; (ii) the construction, using a special algorithm, of the richest dominance relationship; (iii) the application of the Weak Condorcet winner and loser rules to obtain the final ratings, i.e., the optimistic and pessimistic extensions. On the other hand, the methodology proposed in this paper allows the derivation of the LRS scores from the relative preferences in a very straightforward way, requiring only a single run of a standard linear regression software. Third, the number of categories in the rating system derived in Hammer et al. (2006) was exogenously determined (i.e., the decision-maker cannot specify this number a priori), while the approach developed in this paper has the ability to generate a rating system that has the granularity desired by the user of ratings. The derivation of such rating system can be done by solving a mixed-integer programming problem (MIP) which provides a rating system that has the exact number of rating categories specified by the decision-maker and that minimizes the disagreements between the original LRS rating scores and the ratings of the learned system. Fourth, the identification of discrepancies between the derived ratings and those of the learned rating system can be done, with the approach developed in this paper, using widely accepted straightforward statistical methodology, while an intricate combinatorial method was used in Hammer et al. (2006) to reach the same objective. Thus, the two methodologies are qualitatively different, and the methodology of this paper is significantly less involved computationally.

Even though the approach of this study is qualitatively different from Hammer et al. (2006), the results are in surprisingly close agreement, which reinforces greatly their validity. The accuracy of the proposed model's predictions, measured by its correlation coefficient with the S&P ratings, and confirmed by *k*-folding cross-validation, exceeds 95%.

The stability of the constructed non-recursive model is shown in three ways: by the correlation of the predictions with those of other agencies (Moody's and The Institutional Investor (see Appendix)), by predicting 1999 ratings using the non-recursive models derived from the 1998 dataset applied to the 1999 data (showing temporal stability), and by successfully predicting the ratings of several previously non-rated countries.

An additional benefit of this study is that it allows to evaluate the importance of variables. In particular, it shows the importance of including, in addition to economic variables, also political variables (in particular "political stability"), and by identifying "financial depth and efficiency" as a new critical economic indicator of countries' creditworthiness.

The paper is structured as follows. Section 2 describes the data considered and selected for use in this study. We provide a thorough literature review (see references in Tables 2 and 3) and describe the selection of explanatory variables. In Section 3, we provide an overview of a combinatorial-logical technique, the *logical analysis of data* (LAD) (Hammer, 1986), which we use in developing a new methodology for reverse-engineering country risk ratings in Section 4. Subsection 4.1 explains how the patterns of the LAD model are used to compute a discriminant $\Delta(P_{ij})$, called *relative preference,* for each pseudo-observation. The value of $\Delta(P_{ij})$ indicates whether country *i* should be rated higher or lower than country *j*. Relying on the assumption that the $\Delta(P_{ij})$ values provide good approximations of the differences of the ratings, the relative preferences are used to derive an approximation of the ratings called *logical rating scores* (LRS) of countries, calculated using multiple linear regression (Subsection 4.2). Subsection 4.3 describes the method and provides the formulation of the MINLP problem that allows to construct a rating system based the LRS scores, that has any desired granularity level. Section 5 is devoted to a thorough analysis of the results provided by the LRS model. Section 6 shows the temporal validity and the extendability of the proposed rating system. In Section 7, the robustness of the model is confirmed by jackknife cross-validation. Section 8 discusses the predictive power of the variables, and Section 9 presents general conclusions of this study.

2 Data

2.1 Sources

Standard & Poor’s provides country ratings for local and foreign currency debt (Trevino and Thomas, 2001). In this paper, we use the foreign currency country ratings. Countries are more vulnerable to foreign currency obligations. An obligor's capacity to repay foreign currency obligations may be lower than its capacity to repay obligations in its local currency, owing to the sovereign government's relatively lower capacity to repay external versus domestic debt. As noted by Cantor and Packer (1996), foreign currency ratings remain the decisive factor in the international bond market. Indeed, foreign currency obligations are more likely to be acquired by international investors than domestic obligations. Foreign currency ratings reflect economic factors, as well as the country intervention risk, i.e. the risk of a country imposing, for example, exchange controls or a debt moratorium, while local currency ratings exclude country intervention risk.

Within the S&P’s nomenclature, countries which are assigned a label inferior to BB+ are considered as non-investment grade (speculative) countries, and those rated CCC+ or lower are regarded as presenting serious default risks. Ratings labeled from AA to CCC can be modified by the addition of a plus or minus sign to show relative standing within the major rating categories. We consider such subcategories as separate ratings in our analysis.

We have converted the S&P rating scale (from AAA to SD) into a numerical scale (from 21 to 0) and shall liberally refer to both of them as S&P ratings. This conversion is commonly used in the literature, see e.g., Bouchet et al. (2003), Estrella (2000), Ferri et al. (1999), Monfort, Mulder (2000), Hu et al. (2002), Sy (2003). Moreover, Bloomberg, a major provider of financial data services, developed a standard cardinal scale for comparing Moody's, S&P and Fitch-BCA ratings (Kaminsky and Schmukler, 2002); in this scales, a higher numerical value denotes a higher probability of default.

We use the S&P foreign currency country ratings of 69 countries published at the end of December 1998 (S&P, 2001). As mentioned above, country risk ratings encompass economic, financial and political aspects. The statistical data of the economic and financial variables considered in this paper come from Monetary Fund (World Economic Outlook database, 2001), from the World Bank (World Development Indicators database, 2000) while those about the ratio of debt to gross domestic product come from Moody's (2001). Values of political variables are provided by Kaufmann et al. (1999).

2.2 Variable selection criteria

As underlined by Bilson et al. (2001), the selection of variables lends itself to criticism due to the subjectivity and arbitrariness involved in this process. In this paper, the selection of relevant variables is based on three criteria.

The first criterion is the significance of variables for estimating a country's creditworthiness. We have performed an extensive literature review which played an important role in defining the set of candidate variables for inclusion in our model. Tables 2 and 3 list variables that have been considered in the existing literature on country risk. The second criterion is the availability of complete and reliable statistics. We want to avoid difficulties related to missing data that could reduce the significance and the scope of our analysis. The third criterion is the uniformity of data across countries. We have considered, for example, incorporating the unemployment rate statistics disclosed by the World Bank. However, the World Bank underlines that unemployment is compiled according to different definitions: the treatment reserved to temporarily laid off workers, to those looking for their first job, and the criteria referred to for being considered as unemployed, differ significantly between countries.

It is worth noting that in addition to the variables listed in Tables 2 and 3, Haque et al. (1996), Cantor and Packer (1996), Larrain et al. (1997), Monfort and Mulder (2000) and Hu et al. (2002) use a dummy variable that represents the historical solvency of a country. Haque et al. (1996) use the lagged rating at time *(t-1)* as an independent variable in their regression model. Monfort and Mulder (2000) claim that membership in the OECD is likely to be a significant indicator for country risk ratings. The same authors emphasize also the importance of the location of countries, by adding to their set of independent variables two dummy variables to characterize the country's location in Asia or in Latin America. Hu et al. (2002) also use regional dummy variables.

2.3 Selected variables and dataset content

Based on the criteria of relevance, availability and uniformity described above, we have decided to incorporate the following variables[4] in our model: *Gross domestic product per capita*[5] *(GDPc), Inflation rate (IR), Trade balance (TB), Exports' growth rate (EGR), International reserves (RES), Fiscal balance (FB), Debt to GDP (DGDP), Political stability (PS), Government effectiveness (GE), Corruption (COR), Exchange rate (ER), Financial depth and efficiency (FDE)* (see Appendix for a detailed description of these variables and the discussion of their impact on country risk ratings).

While eight of our nine economic variables have been used in the country credit risk rating literature, we include a new variable called *Financial depth and efficiency (FDE)* which is measured as the ratio of the domestic credit provided by the banking sector to the GDP. Households accumulate claims on financial institutions that, acting as intermediaries, pass funds to final users. Correlated to the development of the economy, the indirect lending by savers to investors becomes more efficient and gradually increases assets relative to the GDP. From this perspective, the ratio of domestic credit to the GDP reflects the financial depth and efficiency of the country's financial system. By financial depth, we mean the supply of funds available to the government and private sector of a country. It measures the growth of the banking system since it reflects the extent to which savings are financial.

To our knowledge, the financial depth and efficiency variable has not been considered previously in the evaluation of risk ratings. The reason for us to include this variable is that it captures some information that is relevant to the assessment of the creditworthiness of a country and that is not accounted for by other variables and in particular by the fiscal balance of a country. This statement is based on the Brazilian and Argentine crisis. Many observers attribute the crisis in Brazil and Argentina to their deficient debt and fiscal policies. As noted by Caballero and Krishnamurthy (2001, 2004), this raises the question of why Belgium and Italy which had huge fiscal deficits before the Maastricht Treaty and accumulated debts (more than 100% of the GDP) way beyond those of Argentina did not experience crises similar to those two South American countries, and why Brazil and Argentina were not able to continue their expansionary fiscal policy during downturns the way Belgium or Italy did.

The reason advanced by Caballero and Krishnamurthy (2001, 2004) to explain this is that the financial depth, i.e. the ratio of credit to the private sector over GDP, of Argentina and Brazil was equal to 25% and 30%, respectively, while this ratio exceeded 70% in Italy (late 1990s). This lack of domestic financial depth is exacerbated by the fact that investing in an emerging market requires far more expertise than investing in an advanced one (e.g., acquaintance with political risk, exchange rate risk, and the degree and form of corporate, judicial and government corruption). Only a small set of investors have this

[4] Acronyms in parentheses are used in tables and appendices for referring to variables.
[5] Calculated on the basis of purchasing power parity in international dollars.

expertise, which implies that the financial depth of a country is limited by the liquidity controlled by these specialists.

In view of the considerations described above, we have constructed a dataset involving nine economic/financial variables, and three political variables. We have used the values taken by these variables at the end of 1998. We have compiled the values of these twelve variables for the sixty-nine countries considered: 24 industrialized countries, 11 Eastern European countries, 8 Asian countries, 10 Middle Eastern countries, 15 Latin American countries and South Africa. We use the S&P country risk ratings for these countries at the end of 1998.

2.4 Pairwise comparison of countries: Pseudo-observations

To develop a combinatorial model of country risk ratings, we shall extract from the S&P ratings only the information about the qualitative order relation between countries, but not a quantitative measure of the magnitude of differences between ratings. This order relation between countries will be represented by pseudo-observations constructed as follows.

We associate to every country $i \in I = \{1,\ldots,69\}$ the 13-dimensional vector C_i. The first component of C_i is the country risk rating given by S&P, while the remaining 12 components specify the values of the nine economic/financial and of the three political variables. Instead of considering countries independently of each other, we construct for every pair of countries $i, j \in I$, a *pseudo-observation* P_{ij} providing a comparative description of the two countries.

The pseudo-observations are represented as 13-dimensional vectors. The first component is an indicator which takes the value "1" if the country i in the pseudo-observation P_{ij} has a higher rating (i.e., lower risk) than the country j, takes the value "–1" if the country j has a higher rating than the country i, and takes the value "0" if both countries have the same rating. The other components k , $k = 2,\ldots,13$ of the pseudo-observation $P_{ij}[k]$ are obtained simply by taking the differences of the corresponding components:

$$P_{ij}[k] = C_i[k] - C_i[k], k = 2,\ldots,13 \tag{1}$$

An advantage of this transformation is that it allows us to avoid the problems posed by the fact that the original dataset contains only a small number ($|I|$) of observations. The transformation (8) provides a larger dataset containing $|I|*(|I| -- 1)$ pseudo-observations.

The present study is based on the idea that a risk rating system can be constructed solely from the knowledge of (pre)order of obligors with respect to their creditworthiness. This is in perfect accordance with the general view of the credit risk rating industry. Altman and Rijken (2004) state that the objective of rating agencies is "to provide an accurate relative (i.e., ordinal) ranking of credit risk", which is confirmed by Fitch Ratings (2006) saying that "Credit ratings express risk in relative rank order, which is to say they are ordinal measures of credit". Bhatia (2003) adds that the "rating exercise is highly comparative in nature, based on peer-comparisons"

3 Logical Analysis of Data (LAD) – An overview

The *logical analysis of data* is a combinatorics-, optimization-, and Boolean logic-based methodology for analyzing archives of observations (Boros et al., 1997), and is applicable to data sets in which observations are characterized by binary (Hammer, 1986) as well as numerical (Boros et al., 1997) variables. LAD distinguishes itself from other pattern recognition methods and data mining algorithms by its capacity to discover an irredundant set of variables along with a collection of patterns built on them, in order to explain the positive or negative nature of the observations in an dataset (Boros et al., 2000).

Each observation is represented by an *(n+1)*-dimensional vector, the first component of which is the classification of the observation, the *n* other variables being the inputs. The classification is binary (0 or 1), i.e., it specifies the positive (1) or negative (0) nature of the observation.

The purpose of LAD is to discover a function *f* depending on the *n* input variables (or an approximation of *f)* , allowing the correct discrimination between positive and negative observations. The derived function takes the form of a weighed sum of patterns. *Positive (negative) patterns* are combinatorial rules which impose upper and lower bounds on the values of a subset of input variables, such that a sufficiently high proportion of the positive (negative) observations in the dataset satisfy the conditions imposed by the pattern, and a sufficiently high proportion of the negative (positive) observations violate at least one of the conditions of the pattern.

The conditions defining a pattern specify that the values of some of the variables are "large" or are "small"; more precisely, these conditions require these values to be above or below certain specified levels, called *cutpoints*. By associating an indicator variable to each cutpoint, the dataset is "binarized", i.e., each original numerical variable is replaced by several binary ones.

The following terminology will be useful. The *degree* of a pattern is the number of variables the values of which are bounded in the definition of the pattern. The *prevalence* of a positive (negative) pattern is the proportion of positive (negative) observations covered by it. The *homogeneity* of a positive (negative) pattern is the proportion of positive (negative) observations among those covered by it.

The first step in applying LAD to the dataset is to generate the *pandect*. The number of patterns contained in the pandect of a dataset of such dimensions can be exponentially large, in the order of hundreds of thousands, possibly millions. Because of the enormous redundancy in this set, we shall impose a number of limitations on the set of patterns to be generated, by restricting their degrees (to low values), their prevalences (to high values), and their homogeneities (to high values). The quality of patterns satisfying these conditions is usually much higher than that of patterns having high degrees, or low prevalences, or low homogeneities. Algorithms have been developed for the efficient generation of substantial subsets of the pandect corresponding to reasonable values of the control parameters.

The redundancy among the patterns of the pandect makes necessary the extraction of (usually small) subsets of positive and negative patterns, sufficient for classifying the observations in the dataset. Such collections of positive and negative patterns are called *models.* A model is supposed to contain positive (negative) patterns covering (i.e., whose conditions are satisfied by) each of the positive (negative) observations in the dataset. Furthermore, good models tend to minimize the number of points in the dataset covered by both positive and negative patterns in the model.

The way a LAD model can be used for classification is the following. An observation (whether it is contained or not in the given dataset) which satisfies the conditions of some of the positive (negative) patterns in the model, but which does not satisfy the conditions of any of the negative (positive) patterns in the model, is classified as *positive (negative)*. An observation satisfying both positive and negative patterns in the model is classified with the help of a discriminant which assigns specific weights to the patterns in the model (Boros et al., 2000). More precisely, if p and q represent the number of positive and negative patterns in a model, and if h and k represent the numbers of positive, respectively negative patterns in the model covering a new observation θ, then the value of the discriminant $\Delta(\theta)$ is simply

$$\Delta(\theta) = h/p - k/q, \tag{2}$$

and the corresponding classification is determined by the sign of this expression. Finally, an observation for which $\Delta(\theta) = 0$ is left *unclassified,* since the model either does not provide enough evidence, or provides conflicting evidence; fortunately it has been seen in all the real-life problems considered that the number of unclassified observations is extremely small.

4 Logical rating scores (LRS)

In order to "learn" the S&P rating system, we shall proceed in three steps. In the first step, the proposed model is derived using only the information indicating for each pair of countries with different S&P ratings which one of the two is rated higher. From this information, we shall derive an LAD model, whose discriminant provides a numerical measure $\Delta(P_{ij})$ of the "superiority" of the country i's rating over that of country j. In the second step, applying multiple linear regression we derive new numerical ratings of all countries, called "logical rating scores" (LRS); the logical rating scores are such that their pairwise differences provide the best approximation of the numerical measures Δ obtained in the first step. In the third step, we propose a method to generate, on the basis of the LRS scores, a rating system that comprises a discrete, endogenously defined number of rating categories and is such that the number of countries whose LRS score must be adjusted in order to be in agreement with the learned rating system is minimized. The rating system is obtained through the solution of a mixed-integer programming problem.

4.1 From pseudo-observations to relative preferences

The "observations" of the dataset used in the first step are those pseudo-observations P_{ij}, which correspond to countries i and j having different ratings. Each pseudo-observation P_{ij} is classified as positive or negative, according to the value of the indicator variable, i.e., depending on whether i is rated higher than j or vice versa. Clearly, the training set is anti-symmetric.

After having constructed the LAD model, we compute (according to (9)) the discriminant $\Delta(P_{ij})$ for each pseudo-observation P_{ij} ($i \neq j$). The values $\Delta(P_{ij})$ of the discriminant are called the *relative preferences,* and the [69 x 69]-dimensional anti-symmetric matrix Δ having them as components will be called the *relative preference matrix*. While the LAD model was derived using only those pseudo-observations P_{ij} for which i and j were rated differently, the discriminant matrix components are the values $\Delta(P_{ij})$ ($i \neq j$) taken by the discriminant for every pair of countries, including those that have the same S&P ratings.

It was shown (Hammer et al., 2006) that the reliance on the interpretation of the sign of the relative preferences as an indicator of rating superiority (i.e., a large positive value $\Delta(P_{ij})$ could be interpreted as country i being more creditworthy than country j, while the opposite conclusion could be drawn from a large negative value of the relative preference $\Delta(P_{ij})$) would violate the transitivity requirement of an order relation. In the next section, we therefore relax the overly constrained search for country ratings whose pairwise orderings are in *precise* agreement with the signs of relative preferences, to the more flexible search for *logical rating scores* (*LRS*) having numerical values, whose pairwise differences *approximate* well the relative preferences.

4.2 From relative preferences to logical rating scores using regression analysis

It has been common practice in the research literature (see e.g., Ferri et al., 1999, Hu et al., 2002, Monfort and Mulder, 2000, Sy, 2003) to interpret sovereign ratings as cardinal values. Assuming that the sovereign ratings β can be interpreted as cardinal values, it is natural to view the relative preferences Δ as differences of the corresponding ratings:

$$\Delta(P_{ij}) = \beta_i - \beta_j, \text{for all } i, j \in I, i \neq j \qquad (3)$$

Obviously, the system (3) may or may not be consistent. We shall therefore replace it by:

$$\Delta(P_{ij}) = \beta_i - \beta_j + \varepsilon_{ij}, \text{for all } i, j \in I, i \neq j \qquad (4)$$

The determination of those values of the β's which provide the best L_2 approximation of the Δ's can be found as a solution of the following multiple linear regression problem:

$$\Delta(\pi) = \sum_{k \in I} \beta_k * x_k(\pi) + \varepsilon(\pi) \quad , \qquad (5)$$

where

$$\pi = \{(i, j) \,|\, i, j \in I, i \neq j\} \tag{6}$$

and

$$x_k(i, j) = \begin{cases} 1, & \text{for } k = i \\ -1, & \text{for } k = j \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

4.3 LRS-Based Rating System with Variable Granularity

In order to map the numerical values of the LRS scores to the discrete number of categories in a rating system (S&P's country risk ratings are split into 21 categories), we shall partition the interval of the LRS score values into 21 sub-intervals corresponding to the twenty one rating categories used by S&P's. The partitioning is defined with respect to *cutpoints* x_i such that $x_0 \leq x_1 \leq \ldots < x_j \leq \ldots \leq x_{20} \leq x_{21}$ where j indexes the rating categories $(j=1,2,\ldots,21$ with 1 corresponding to "selected default" (SD), i.e., the weakest rating category and 21 corresponding to AAA, i.e., the best rating category).

In reality, such a partitioning may not exist. Therefore, in order to take "noisiness" into account, we shall replace the LRS scores β_k of country k by the *adjusted LRS scores* δ_k, and find values of δ_k for which such a partitioning exists and in which the number of countries for which an adjustment of the LRS score is necessary is minimized. This objective shall be achieved by solving the mixed-integer programming problem presented below in which the following notations are used. The set of countries (rating categories) is denoted by N (J), $|J|$ represents the cardinality (i.e., the number of rating categories) of the set J and $j(k)$ is the S&P's rating category of country k. The parameters $\overline{\beta} = \max_{k \in N} \beta_k$ $(\underline{\beta} = \min_{k \in N} \beta_k)$ indicate the highest (smallest) LRS scores assigned to a country, while M and ε respectively represent a large and an infinitesimal positive numbers. Regarding the decision variables, δ_k is the adjusted LRS score of country k, x_j is a cutpoint associated with the S&P's rating category j and α_k is a binary decision variable taking value 1 if the LRS score of country k needs adjusting, and taking value 0 otherwise.

$$
\begin{array}{lll}
\textit{minimize} & \sum_{k \in I} \alpha_k & \\
\textit{subject to} & \delta_k \leq x_{j(k)} & k \in I \\
 & x_{j(k)-1} + \varepsilon \leq \delta_k & k \in I \\
 & \delta_k - \beta_k \leq M\alpha_k & k \in I \\
 & \beta_k - \delta_k \geq M\alpha_k & k \in I \\
 & \underline{\beta} = x_0 \leq x_1 \leq x_2, \ldots, \leq x_j, \ldots, x_{20} \leq x_{21} = \overline{\beta} & \\
 & \underline{\beta} \leq \delta_k \leq \overline{\beta} & k \in I \\
 & \alpha_k \in \{0,1\} & k \in I
\end{array} \tag{8}
$$

The objective function minimizes the number of countries for which an LRS score adjustment is necessary. Ideally (since the objective here is to reconstruct the S&P's rating system), a country k should be rated j if its LRS score is between x_{j-1} and x_j (e.g., it should be rated AAA if its LRS score is between x_{20} and x_{21}). The first two sets of constraints allow δ_k to be equal to β_k if this is the case and require an adjustment of the LRS score (i.e., $\delta_k \neq \beta_k$) if the above conditions are not satisfied. The third and fourth sets of constraints define the decision variables α_k and force each of them to take value 1 if an LRS adjustment is needed (i.e., $\delta_k \neq \beta_k$) for the country k they are associated with. By setting $M = \left(\overline{\beta} - \underline{\beta}\right)^2$ (i.e., the largest possible adjustment), we guarantee that the constraints can always be satisfied if $\alpha_k = 1$. The fifth set of constraints ensures that the value taken by the cutpoint is an increasing function of the quality of the rating. The dimensions of the MIP problem depend on the number of countries and the number of categories of the learned rating system. More precisely, the above MIP problem contains (4*|I| + |J|) constraints and involves |I| binary and (|I| + |J| - 1) continuous decision variables.

Denoting by $\left(x_j^*, \alpha_k^*, \delta_k^*\right)$ the optimal solution of problem (8), the LRS-based rating R_k of country k is defined as follows:

$$
R_k = j \text{ if } \begin{cases} \beta_k \leq x_j^* \\ \beta_k > x_{(j-1)}^* \end{cases}, \quad k \in N, j = 1, \ldots, 21 \tag{9}
$$

5 Evaluation of the results

The estimation of the regression model (6) used in deriving LRS shows that its statistical significance is very high. Indeed, its *p-value* turns out to be less than 0.0005 and its R-square is 95.2%.

In order to evaluate the results obtained using the LRS model, we shall carry out several comparisons. We shall compare LRS with the S&P ratings, as well as with the results obtained using the non-recursive regression model. We shall further compare LRS with the scores associated to Moody's ratings at the end of December 1998, and with those provided by The Institutional Investor in March 1999. In this analysis, we shall evaluate first the relative preferences obtained using LAD, and then the LRS derived from them.

5.1 Evaluation of relative preferences

In this section, we compare the canonical relative preferences d^{LRS}_{ij} associated with the logical rating scores to those ($d^{S\&P}_{ij}$, d^{M}_{ij} , d^{II}_{ij}) obtained respectively from the scores associated with S&P's ratings, Moody's ratings, and The Institutional Investor's scores. The canonical relative preferences d_{ij} associated to any pair of countries i and j are computed as $d_{ij} = s_i - s_j$. where s_i denotes the numerical scores corresponding to the sovereign ratings of country i. It appears that the canonical preferences of the logical rating scores are in extremely high agreement not only with the S&P's canonical relative preferences, but also with those of the other rating agencies. Indeed, the correlation level between the LRS canonical relative preferences and those derived from the scores of the rating agencies range between 94.11% and 95.54%.

5.2 Evaluation of logical rating scores

We shall analyze now the correlation levels between the logical rating scores and the scores associated with the ratings of S&P, Moody and The Institutional Investor. The very high levels of correlation between the scores show that the logical rating scores are very good approximations of the S&P ratings, as well as of those provided by other rating agencies.

5.3 Discrepancies between S&P's and LRS and their resolution

It has been seen in the previous section that the LRS scores and the S&P ratings are in close agreement. However, since the logical rating scores and the S&P's ratings are not expressed on the same scale, the comparison of the two scores of an individual country presents a challenge. In order to overcome this difficulty, we shall apply a linear transformation to the LRS, which brings them to the same scale as the S&P's ratings. This is accomplished by determining the coefficients a and c for the transformation $a*\beta_i + c$ of the LRS β_i in such a way that the mean square difference between the transformed LRS and the S&P's ratings is minimized. As a result of this transformation, the LRS become directly comparable with the S&P's ratings. Clearly, the consistency of the LRS and S&P's ratings is not affected by this transformation.

Using formula (11), we then compute the confidence intervals for the transformed LRS of each country. In 1998, five countries have a S&P's rating that does nor fall within the confidence interval of the transformed LRS. Columbia appears to be too favorably rated by S&P's, while Hong-Kong, Malaysia, Pakistan and Russia appear to be rated too harshly by S&P's. The evolution of the S&P's ratings for Columbia, Pakistan and Russia is in agreement with the 1998 LRS of these countries, and underlines the prediction capability of the LRS model. It is remarkable that the evolution of the S&P's ratings of Malaysia and Hong-Kong is also in agreement with their 1998 LRS. Indeed, both Malaysia and Hong-Kong have been upgraded shortly thereafter, the former moving from BBB- to BBB in November 1999, and the latter from A to A+ in February 2001.

5.4 Discrepancies between S&P's ratings and LRS-based ratings

In this section, we generate two rating systems. The first one contains the same number (21) of rating categories as the S&P's rating systems. The second one assigns each country to one of the three main types of categories of borrowers (investment-, speculative- and default grade) commonly identified by rating agencies. The construction of the two rating systems is performed using the open-source MIP solver Cbc (COIN-OR, 2008) which solves both MIP problems to optimality in less than 5 seconds of CPU time.

For the most granular rating system (i.e., with 21 rating categories), the LRS-based ratings of 21 countries must be slightly modified. For 18 of those countries, a one-notch adjustment is required, while for the other three (France, Japan, Colombia), a 2-notch adjustment is needed.

For the second model, 65 countries (94.2%) receive the same LRS-based ratings as the S&P's ratings. The four discrepancies concern:

- Latvia, Lithuania, Colombia which are considered as investment-grade by S&P's (i.e., they are all rated BBB- which is the lowest rating category receiving the label investment-grade) and which are rated as speculative-grade with the LRS-based ratings;
- India considered as default-grade (speculative-grade) by the S&P's (LRS-based) ratings.

We observe here the predictive power of the LRS-based ratings: Columbia was shortly afterwards downgraded to a speculative rating by S&P's, while India was promoted to a speculative-grade rating within one year by S&P's.

6 Extendability of the LRS model

The LRS model makes possible the calculation of logical rating scores on the basis of new data. Indeed, in order to obtain logical rating scores for the new data all that has to be done is to recalculate the relative preferences using the existing LAD discriminant, and then to rerun linear regression in order to "reintegrate" the new relative preferences into the new logical rating scores. Clearly, the analysis of new data by the LRS model is carried out independently of any possible new ratings of S&P's, since the LAD discriminant underlying the LRS model has already been constructed and remains valid. The remarkable feature of the LRS model is that for important cases of data perturbation, including those resulting from temporal changes in the values of the independent variables, or the addition of previously non-rated countries, the new logical rating scores maintain to a large extent their validity.

6.1 Temporal changes in data

In this section, we shall construct logical rating scores based on the 1999 data, in order to evaluate the robustness and consistency of the LRS model. This goal is accomplished by comparing the LRS scores built on the 1998 S&P's ratings with S&P's 1999 ratings. The very high (94.12%) correlation level between the LRS canonical relative preferences and the canonical relative preferences associated with the S&P's ratings attest to the very strong temporal stability of the LRS model..

6.2 Discrepancies and their resolution

In order to identify the discrepancies, we have to recalculate the prediction confidence intervals for the new, 1999 observations, and thus for the transformed LRS of each country.

Let us introduce some notations. Let n and p refer to the number of observations and predictors, respectively. The expression *t(1- α/2,n-p)* refers to the Student test with *(n-p)* degrees of freedom, and with upper and lower tail areas of α/2. Let X_j be the p-dimensional vector of the values taken by the observation Y_j on the p predictors, while $X_p^{'}$ be the transposed of X_j. Let the expression $(X'X)^{-1}$ refer to the variance-covariance matrix, i.e. the inverse of the *[pxp]*-dimensional matrix *(X'X)*. Denoting by MSE the mean square of errors, the estimated variance $s^2[\hat{Y_j}]$ of the predicted rating is:

$$s^2[pred] = MSE * [1 + X_j^{'}(X'X)^{-1} X_j] \quad , \tag{10}$$

while the $(1-\alpha)$ confidence interval for $\hat{Y}_{j,n}$ will be given by:

$$\{\hat{Y}_{j,n} - t(1-\alpha/2, n-p) * s[pred], \hat{Y}_{j,n} + t(1-\alpha/2, n-p) * s[pred]\} \tag{11}$$

We say that there is a discrepancy between S&P's rating R_j^{SP} and the logical rating score if:

$$R_j^{SP} \notin \{\hat{Y}_{j,n} - t(1-\alpha/2, n-p) * s[pred], \hat{Y}_{j,n} + t(1-\alpha/2, n-p) * s[pred]\} \text{ for } \alpha = 0.1 \tag{12}$$

Applying the 1998 LRS model to the 1999 data, only two countries (Russia and Hong-Kong) have S&P's ratings that are outside the confidence intervals of the corresponding transformed LRS. These two countries appear to be rated too harshly by S&P's, while the evolution of the S&P's ratings for Hong-Kong and Russia is in agreement with the 1999 LRS of these countries.

6.3 Discrepancies between S&P's ratings and LRS-based ratings

As previously, we generated two rating systems containing respectively 21 and 3 rating categories. The corresponding MIP problems were also solved to optimality in a few seconds.

The reconstruction of the 21-category rating system requires a one-notch rating adjustment for 19 countries and a two-notch adjustment is needed for 3 countries. Regarding the 3-category rating system, two of the discrepancies (Columbia and India) observed in 1998 were resolved by the S&P's rating modifications that took place in 1999. The other two discrepancies (Latvia and Lithuania) remain.

6.4 LRS ratings of countries previously not rated by S&P's

The availability of the LAD discriminant, which does not involve in any way the previous years' S&P's ratings, makes it possible to rate previously not rated countries in the following way. After calculating first the attribute values of all the pseudo-observations involving the new countries to be evaluated, the relative preferences are to be calculated for these pseudo-observations, and the resulting columns and rows have to be added to the matrix of relative preferences. The new LRS for all the countries (new and old) should then be determined by running the multiple linear regression model (5).

In order to evaluate the capability of LRS to correctly predict S&P's ratings, we compare the LRS predicted as described above, with the S&P's ratings when they first become available. The direct comparison between these ratings is carried out using the linear transformation described above.

Using formula (11), we compute the confidence intervals for the transformed LRS of four countries never rated by S&P's by December 1998. Our predictions for three Guatemala, Jamaica and Papua New Guinea correspond perfectly to the first time (subsequent) S&P's ratings. The comparison between the LRS and the first S&P's rating (SD) given in July 2000 to Ecuador shows that S&P's rated it too harshly, since one month later S&P's raised its rating to B-, thus justifying the LRS prediction.

7 Cross-validation of relative preferences

Since the role of the LAD discriminant is to capture the rules of country creditworthiness, which are implicit in the S&P's ratings, it is the most important component of the LRS methodology. As any learning procedure, LAD can be susceptible to overfitting; if this happens, the performance of the resulting model can be excellent on the training data, but can perform very poorly on new observations.

To test whether the LAD discriminant is affected by overfitting, we shall use the commonly used statistical technique of cross-validation. The particular type of cross-validation technique used here is known as "jackknife" (Quenouille, 1949) or "leave-one-out". In broad terms, the jackknife technique consists of removing from the dataset one observation at a time, learning a model from all the remaining observations, evaluating the resulting model on the removed observation; and then repeating these steps for each observation in the dataset. If the predicted evaluations are "close to" the actual values of the observations, as well as in-the-sample evaluations, then the model is not affected by overfitting.

The correlation levels presented in Table 1 indicate that the matrices of relative preferences Δ^{JK} and d_{JK}^{LRS} obtained through the jackknife procedure are highly correlated with (i) the canonical relative preferences corresponding to the S&P's ratings $d^{S\&P}$, and (ii) the in-the-sample relative preferences Δ, as well as the logical rating scores d^{LRS} . This shows that Δ and d^{LRS} are not affected by overfitting.

Table 1: Correlation between cross-validated relative preferences

	$d^{S\&P}$	Δ	d^{LRS}	Δ^{JK}	d_{JK}^{LRS}
$d^{S\&P}$	100%	93.21%	95.54%	92.98%	95.26%
Δ	93.21%	100%	97.57%	96.48%	96.36%
d^{LRS}	95.54%	97.57%	100%	96.35%	96.89%
Δ^{JK}	92.98%	96.48%	96.35%	100%	97.43%
d_{JK}^{LRS}	95.26%	96.36%	96.89%	97.43%	100%

8 Importance of variables

The methodology developed in this paper allows to evaluate the importance of variables in rating countries' creditworthiness. In LAD, the importance of variables is associated with their participation in the patterns of the discriminant, and is usually measured by the fraction of patterns containing a particular variable. The patterns of the 1998 LAD model show that the three variables that appear most frequently in the patterns of the LAD discriminant are financial depth and efficiency, political stability, and gross domestic product per capita (appearing in 47.5%, 39.4% and 35.6% of the LAD patterns, respectively).

The fact that a political variable appears among the three most significant ones in the selected set provides additional justification for the inclusion of political variables in country risk rating models. This result is in agreement with the cost-benefit approach to country risk (i.e., the risk of defaulting is heavily impacted by the political environment, see Brewer, Rivoli, 1990, 1997, Citron, Neckelburg, 1987, Mauro, 1993, Lee 1993, Afonso, 2003, Manasse et al., 2003) which is not shared by some economists at the IMF (Haque et al., 1996, 1998). The finding that political stability happens to be the most important political variable may provide valuable insight for sovereign credit risk related decisions.

The fact that the LAD approach identifies gross domestic product per capita as significant is not surprising since most studies on country risk ratings acknowledge its key importance in evaluating the solvency of a country. An interesting new result is that it identifies the *financial depth and efficiency (FDE)* variable as a crucial factor impacting country risk.

9 Concluding Remarks

The central objective of this paper was to develop a transparent, self-contained, consistent, and stable country risk rating system, closely approximating the country risk ratings provided by a major rating agency (S&P's). We proposed a model that achieves the stated objectives, using the combinatorial-logical technique of Logical Analysis of Data.

The proposed model is highly *accurate*, having a 95.5% correlation level with the actual S&P's ratings and almost equally high correlations with Moody's and The Institutional Investor. The model avoids overfitting, as demonstrated by the 99.1% correlation between in- and out-of-sample rating predictions, calculated by *k*-fold cross-validation. The proposed model is *transparent* since it makes the role of the economic-financial and political variables explicit.

The proposed model is distinguished from the rating models in the existing literature by its *self-contained* nature, i.e., by its non-reliance on any information derived from lagged ratings. Therefore, the high level of correlation between predicted and actual ratings cannot be attributed to the reliance on lagged ratings, and is a reflection of the relevance and predictive power of the independent variables included in these models. The significant advantage of the non-recursive nature of the proposed model is their applicability to not-yet-rated countries.

The *consistency* of the proposed model is illustrated by the fact that the few discrepancies between the S&P's ratings and those of the proposed model were resolved by subsequent changes in S&P's ratings. The *stability* of the constructed non-recursive model is shown in two ways: by predicting 1999 ratings using the non-recursive model derived from the 1998 dataset, and – most importantly – by successfully predicting the ratings of several previously non-rated countries, i.e., in full agreement with subsequent ratings of those countries by S&P.

Not only does the proposed model generate the LRS numerical scores, it also allows for the construction of a discrete rating system which comprises the number of rating categories specified by the user of the model and that is in close agreement with the learned system. The ability to generate a prescribed number of rating categories is an important advantage compared with the approach developed in Hammer et al. (2006) where the number of rating categories was not under control.

The study provides new insights on the importance of variables by supporting the necessity of including, in addition to economic variables, also political ones (in particular "political stability"), and by identifying "financial depth and efficiency" as a new critical factor in assessing country risk.

The significance of the results of this paper is further confirmed in a related study (Hammer et al., 2006), in which the LAD discriminant, constructed in the same way as in this paper, is used for deriving a partial order representing the creditworthiness of countries, and then extending this order to a new rating system. The comparison of the results of these two studies shows that – in spite of a different data analysis approach – the predicted ratings are strongly correlated, at the surprising level of 98%.

References

Afonso A. 2003. Understanding the Determinants of Sovereign Debt Ratings: Evidence for the Two Leading Agencies. *Journal of Economics and Finance* **27** (1), 56-74.

Afonso A., Gomes P., Rother P. 2007. What "Hides" behind Sovereign Debt Ratings? *European Central Bank Working Paper Series* **711**.

Altman E.I., Rijken H.A. 2004. How Rating Agencies Achieve Rating Stability. *Journal of Banking and Finance* **28**, 2679-2714.

Altman E.I., Saunders A. 1997. Credit Risk Measurement: Developments over the Last 20 Years. *Journal of Banking and Finance* **21** (11-12), 1721-1742.

Aylward L., Thorne R. 1998. Countries' Repayments Performance Vis-à-Vis the IMF", *IMF Staff Papers* **45** (4), 595-619.

Baek I.M., Bandopadhyaya A., Du C. 2005. Determinants of Market-Assessed Sovereign Risk: Economic Fundamentals or Market Risk Appetite? *Journal of International Money and Finance* **24** (4), 533-548.

Basel Committee on Banking Supervision. 2005. Basel II: Revised International Capital Framework.

Bennell J.A., Crabbe D., Thomas S., Gwilym O. 2006. Modelling Sovereign Credit Ratings: Neural Networks Versus Ordered Probit. *Expert Systems With Applications* **30** (3), 415-425.

Bilson C. M., Brailsford T.J., Hooper V.J. 2001. Selecting Macroeconomic Variables as Explanatory Factors of Emerging Stock Market Returns. *Pacific-Basin Finance Journal* **9** (4), 401-426 .

Bhatia A.V. 2003. Sovereign Credit Risk Ratings: An Evaluation. *IMF Working Paper* ***WP/* 03/170**.

Bissoondoyal-Bheenick E. 2005. An Analysis of the Determinants of Sovereign Ratings. *Global Finance Journal* **15**, 251-280.

Bissoondoyal-Bheenick E., Brooks R., Yip A.Y.N. 2006. Determinants of Sovereign Ratings: A Comparison of Case-Based Reasoning and Ordered Probit Approaches. *Global Finance Journal* **17** (1), 136-154.

Boros E., Hammer P.L., Ibaraki T., Kogan A. 1997. Logical Analysis of Numerical Data, *Mathematical Programming* **79**, 163-190.

Boros E., Hammer P.L., Ibaraki T., Kogan A., Mayoraz E., Muchnik I. 2000. An Implementation of Logical Analysis of Data. *IEEE Transactions on Knowledge and Data Engineering* **12**(2), 292-306.

Bouchet M.H., Clark E., Groslambert B. 2003. Country Risk Assessment: A Guide to Global Investment Strategy. John Wiley & Sons Ltd. Chichester, England.

Bourke P., Shanmugam B. 1990. An Introduction to Bank Lending. Addison-Wesley Business Series.

Brewer T.L., Rivoli P. 1990. Politics and Perceived Country Creditworthiness in International Banking. *Journal of Money, Credit and Banking* **22**, 357-369.

Brewer T.L., Rivoli P. 1997. Political Instability and Country Risk. *Global Finance Journal* **8** (2), 309-321.

Caballero R., Krishnamurthy A. 2001. International and Domestic Collateral Constraints in a Model of Emerging Market Crises. *Journal of Monetary Economics* **48** (3), 513–548.

Caballero R., Krishnamurthy A. 2004. Fiscal Policy and Financial Depth. *National Bureau of Economic Research (NBER) Working Papers* **10532**.

Cantor R. 2004. An Introduction to Recent Research on Credit Risk Rating. *Journal of Banking and Finance* **28**, 2565-2573.

Cantor R., Packer F. 1996. Determinants and Impact of Sovereign Credit Ratings. *FRBNY Economic Policy Review,* 37-53.

Citron J.T., Neckelburg G. 1987. Country Risk and Political Instability. *Journal of Development Economics* **25**, 385-395.

Claessens S., Embrechts G. 2002. Basel II, Sovereign Ratings and Transfer Risk: External versus Internal Ratings. Presentation at the *Basel II: An Economic Assessment, Bank for International Settlements*, Basel.

COIN-OR. 2008. http://www.coin-or.org/

Cook W.D., Hebner K.J. 1993. A Multicriteria Approach to Country Risk Evaluation: With an Example Employing Japanese Data. *International Review of Economics and Finance* **2** (4), 327-348.

Durbin E., Ng D. 2005. The Sovereign Ceiling and Emerging Market Corporate Bond Spreads. *Journal of International Money and Finance* **24** (4), 631-649.

Easton S.T., Rockerbie D.W. 1999. What's in a Default ? Lending to LDCs in the Face of Default Risk. *Journal of Development Economics* **58** (2), 319-332.

Eaton J., Gersovitz M., Stiglitz J.E. 1986, The Pure Theory of Country Risk. *European Economic Review* **30**, 481-513.

Eliasson A. 2002. Sovereign Credit Ratings. *Working Papers 02-1, Deutsche Bank.*

Erb C.B., Harvey C.R., Viskanta T.E. 1996. Expected Returns and Volatility in 135 Countries. *Journal of Portfolio Management*, 46-58.

Estrella A. 2000. Credit Ratings and Complement Sources of Credit Quality Information. *Basel Committee on Banking Supervision Working Paper* **3.**

Feder G., Uy V.U. 1985. The Determinants of International Creditworthiness and their Policy Implications. *Policy Modeling* **7** (1), 133-156.

Ferri G., Liu L-G., Stiglitz J. 1999. The Procyclical Role of Rating Agencies: Evidence from the East Asian Crisis. *Economic Notes* **3**, 335-355.

Fitch Ratings. 2006. The Role of Support and Joint Probability Analysis in Bank Ratings. *Fitch Special Report*.

Hammer P.L. 1986. Partially Defined Boolean Functions and Cause-Effect Relationships. *International Conference on Multi-Attribute Decision Making Via OR-Based Expert Systems*. University of Passau, Passau, Germany.

Hammer P.L., Kogan A., Lejeune M.A. 2006. Modeling Country Risk Ratings Using Partial Orders. *European Journal of Operational Research* **175** (2), 836-859.

Haque N.U., Kumar M.S., Mark N., Mathieson D. 1996. The Economic Content of Indicators of Developing Country Creditworthiness. *International Monetary Fund Working Paper* **43** (4), 688-724.

Haque N.U., Kumar M.S., Mark N., Mathieson D. 1998. The Relative Importance of Political and Economic Variables in Creditworthiness Ratings. *International Monetary Fund Working Paper* **46**, 1-13.

Haque N.U., Mark N., Mathieson D. 1997. Rating the Raters of Country Creditworthiness. *Finance & Development* **34**, 10-13.

Hu Y.-T., Kiesel R., Perraudin W. 2002. The Estimation of Transition Matrices for Sovereign Credit Ratings. *Journal of Banking and Finance* **26** (7), 1383-1406.

The International Monetary Fund. 2001. World Economic Outlook Washington DC.

Jüttner J.D., McCarthy J. 2000. Modeling a Rating Crisis. Macquarie University, Unpublished, Sidney, Australia, http://www.econ.mq.edu.au/staff/djjuttner/SOVEIG2.pdf.

Kaminsky G., Schmukler S.L. 2002. Emerging Market Instability: Do Sovereign Ratings Affect Country Risk and Stock Returns? *World Bank Economic Review* **16**, 171-195.

Kaufmann D., Kraay A., Zoido-Lobaton P. 1999. Governance Matters. *World Bank Policy Research Department Working Paper* **2196**.

Kaufmann D., Kraay A., Zoido-Lobaton P. 2002. Governance Matters II: Updated Indicators for 2000/01. *World Bank Policy Research Department Working Paper* **2772.**

Kunczik M. 2000. Globalization: News Media, Images of Nations and the Flow of International Capital with Special Reference to the Role of Rating Agencies, Paper presented at the IAMCR Conference, Singapore, 1-49.

Larrain G., Reisen H., von Maltzan J. 1997. Emerging Market Risk and Sovereign Credit Ratings. *OECD Development Center*, 1-30.

Lee S.H. 1993. Relative Importance of Political Instability and Economic Variables on Perceived Country Creditworthiness. *Journal of International Business Studies* **24**, 801-812.

Levich R., Reinhart C., Majnoni G. 2002. *Ratings, Rating Agencies and the Global Financial System.* Kluwer Academic Press. New York.

Manasse P., Roubini N., Schimmelpfennig A. 2003. Predicting Sovereign Debt Crises. *IMF Working Paper* ***WP/*** **03/221**.

Mauro P. 1993. Essays on Country Risk, Asset Markets and Economic Growth. Harvard University Dissertation.

Monfort B., Mulder C. 2000. Using Credit Ratings for Capital Requirements on Lending to Emerging Market Economies: Possible Impact of a New Basel Accord. *IMF Working Paper* WP/00/69.

Moody's. 2001. *Moody's Country Credit Statistical Handbook*, First Edition, New York..

Mora N. 2006. Sovereign Credit Ratings: Guilty beyond Reasonable Doubt? *Journal of Banking and Finance* **30 (7),** 2041-2062.

Noy I. 2008. Sovereign Default Risk, the IMF and Creditor Moral Hazard. *Journal of International Financial Markets, Institutions & Money* **18** (1), 64-78.

Quenouille M. 1949. Approximate Tests of Correlation in Time Series. *Journal of the Royal Statistical Society, Series B* **11**, 18-84.

Reinhart C.M. 2002. Default, Currency Crises, and Sovereign Credit Ratings. *World Bank Economic Review* **16**, 151-170.

Shao J., Tu D. 1995. *The Jackknife and Bootstrap*. New York, Springer-Verlag.

Standard & Poor's. 2001. *Sovereign Ratings History since 1975.* New York.

Standard & Poor's. 2007. *Sovereign Defaults and Rating Transition Data: 2006.* New York, NY.

Sy A.N.R. 2003. Rating the Rating Agencies: Anticipating Currency Crises or Debt Crises? *IMF Working Paper* ***WP/*** **03/122**.

Trevino L., Thomas S. 2001. Local Versus Foreign Currency Ratings: What Determines Sovereign Transfer Risk? *Journal of Fixed Income* **11** (1), 65–76.

Van Gestel T., Baesens B., Van Dijcke P., Garcia J, Suykens J.A.K., Vanthienen J. 2006. A Process Model to Develop an Internal Rating System: Sovereign Credit Ratings. *Decision Support Systems* **42** (2), 1131-1151.

World Bank. 2000. *World Development Indicators 2000.* Washington D.C.

Appendix

Table 2: Economic variables and literature

Variable	Literature
Consumer price index	Hu et al. (2002), Larrain et al. (1997)
Credit claims on central government growth rate	Monfort, Mulder (2000)
Current account balance / GDP	Baek et al. (2005), Bissoondoyal-Bheenick E. (2005), Bissoondoyal-Bheenick et al. (2006), Brewer, Rivoli (1990), Cook, Hebner (1993), Haque et al. (1996, 1998), Larrain et al. (1997)
Debt / exports	Aylward, Thorne (1998), Cantor, Packer (1996), Dailami, Leipziger (1997), Feder, Uy (1985), Hu et al. (2002), Larrain et al. (1997), Lee (1993), Monfort, Mulder (2000)
Debt[6] / GDP	Aylward, Thorne (1998), Brewer, Rivoli (1990), Cook, Hebner (1993), Feder, Uy (1985), Haque et al. (1996, 1998), Hu et al. (2002), Lee (1993)
Debt / reserves	Manasse et al. (2003), Monfort, Mulder (2000)
Dependence on oil exportation	Feder, Uy (1985)
Domestic investment / GDP	Larrain et al. (1997), Monfort, Mulder (2000)
Exports / GDP	Aylward, Thorne (1998), Bissoondoyal-Bheenick E. (2005)
Exports concentration	Feder, Uy (1985)
Exports growth rate	Feder, Uy (1985), Haque et al. (1996, 1998), Monfort, Mulder (2000), Noy (2008)
Exports vulnerability to external shocks	Feder, Uy (1985)
External debt / GDP	Afonso (2003), Baek et al. (2005), Bennell et al. (2006), Bissoondoyal-Bheenick E. (2005), Brewer, Rivoli (1990), Larrain et al. (1997), Manasse et al. (2003), Monfort, Mulder (2000), Noy (2008)
Fiscal balance[7]	Bennell et al. (2006), Bissoondoyal-Bheenick E. (2005), Cantor, Packer (1996), Cook, Hebner (1993), Larrain et al. (1997), Lee (1993), Monfort, Mulder (2000), Noy (2008)
Foreign investment policy	Cook, Hebner (1993), Bissoondoyal-Bheenick et al. (2006)
GDP growth rate	Baek et al. (2005), Bennell et al. (2006), Cantor, Packer (1996), Cook, Hebner (1993), Feder, Uy (1985), Haque et al. (1996,1998), Hu et al.(2002), Larrain et al. (1997), Monfort, Mulder (2000)
GDP per capita	Afonso (2003), Bennell et al. (2006), Bissoondoyal-Bheenick E. (2005), Bissoondoyal-Bheenick et al. (2006), Dailami, Leipziger (1997), Erb et al. (1996), Feder, Uy (1985), Larrain et al. (1997), Monfort, Mulder (2000)
GDP per capita growth rate	Aylward, Thorne (1998), Haque et al. (1996), Lee (1993)
Gross investment / GDP	Easton, Rockerbie (1999)
Imports / GDP	Aylward, Thorne (1998), Haque et al.(1996)
Indicator for economic development	Bennell et al. (2006), Cantor, Packer (1996)
Inflation rate	Aylward, Thorne (1998), Baek et al. (2005), Bennell et al. (2006), Bissoondoyal-Bheenick E. (2005), Bissoondoyal-Bheenick et al. (2006), Cantor, Packer (1996), Dailami, Leipziger (1997), Erb et al. (1996), Haque et al. (1996, 1998), Larrain et al. (1997), Monfort, Mulder (2000), Noy (2008)
International reserves / imports	Aylward, Thorne (1998), Baek et al. (2005), Dailami, Leipziger (1997), Easton, Rockerbie et al. (1999), Feder, Uy (1985), Haque et al. (1996,1998), Hu et al.(2002), Lee (1993), Monfort, Mulder (2000)
Long-term debt / GDP	Easton, Rockerbie et al. (1999)
Real exchange rate	Baek et al. (2005), Bissoondoyal-Bheenick E. (2005), Cook, Hebner (1993),

[6] The word "debt" can encompass foreign, total, debt service or external debt, depending on authors.
[7] Central government spending / GDP, domestic public debt / GDP and are used as a proxy for this variable.

	Haque et al. (1996,1998), Larrain et al.(1997), Monfort, Mulder (2000)
Savings / GDP	Larrain et al. (1997)
Short-term debt / reserves	Dailami, Leipziger (1997)
Short-term debt / total debt	Monfort, Mulder (2000)
Terms of trade	Bissoondoyal-Bheenick et al. (2006), Easton, Rockerbie (1999), Feder, Uy (1985), Haque et al. (1996,1998), Monfort, Mulder (2000)
Trade openness	Easton, Rockerbie (1999)
Treasury bill rate	Haque et al. (1998), Monfort, Mulder (2000)

Table 3: Political variables and literature

Variable	Literature
Anti-governmental demonstrations	Haque et al. (1998)
Armed conflicts (or riots)	Brewer, Rivoli (1990), Cook, Hebner (1993), Haque et al. (1998)
Assassination	Haque et al. (1998)
Corruption	Mauro (1993)
Coups	Haque et al. (1998)
General strikes	Haque et al. (1998)
Guerilla warfare	Haque et al. (1998)
Influence of the middle class	Cook, Hebner (1993), Mauro (1993)
Legal system	Mauro (1993)
Major government crises	Haque et al. (1998)
Political change	Mauro (1993), Brewer, Rivoli (1990)
Political legitimacy	Brewer , Rivoli (1990)
Political stability	Afonso (2003), Brewer, Rivoli (1990, 1997), Citron, Neckelburg (1987), Cook, Hebner (1993), Feder, Uy (1985), Lee (1993), Manasse et al. (2003), Mauro (1993)
Probability of opposition group takeover	Mauro (1993)
Purges	Haque et al. (1998)
Red tape, bureaucracy	Mauro (1993)
Relationships with neighboring countries	Mauro (1993)
Revolutions	Haque et al. (1998)
Social Stability	Cook, Hebner (1993)
Stability of labor	Mauro (1993)
Terrorism	Mauro (1993)

Table 4: Standard & Poor's country risk ratings: average one-year transition rates (1975-2006)

	AAA	AA	A	BBB	BB	B	CCC	SD
AAA	97.9	2.1	0.0	0.0	0.0	0.0	0.0	0.0
AA	3.7	94.1	1.3	0.0	0.5	0.4	0.0	0.0
A	0.0	3.2	94.7	2.1	0.0	0.0	0.0	0.0
BBB	0.0	0.0	8.6	87.4	2.9	1.1	0.0	0.0
BB	0.0	0.0	0.0	6.4	86.1	5.0	1.5	1.1
B	0.0	0.0	0.0	0.0	11.0	83.2	3.9	1.9
CCC	0.0	0.0	0.0	0.0	0.00	17.6	41.2	41.2

Source: Standard & Poor's (2007)

Table 5: Moody's rating system

	Levels	Meaning		Levels	Meaning
INVESTMENT RATING	**Aaa**	Highest quality	SPECULATIVE RATING	**Ba1**	Likely to fulfill obligations
	Aa1	High quality		**Ba2**	
	Aa2			**Ba3**	Ongoing uncertainty
	Aa3			**B1**	High risk obligations
	A1	Strong payment capacity		**B2**	
	A2			**B3**	
	A3		DEFAULT RATING	**Caa**	Current vulnerability to default or in default
	Baa1	Adequate payment capacity		**Ca**	In bankruptcy or default.
	Baa2			**D**	
	Baa3				

Moody's sovereign ratings are defined, as "a measure of the ability and willingness of the country's central bank to make available foreign currency to service debt, including that of central government itself" (Moody's, 1995). Similarly to S&P, Moody's uses a nominal rating scale, which contains the same number of categories as S&P ratings. A large proportion of countries receive the same rating from Moody's and S&P's, and when they are different, the difference is usually not more than one notch.

The Institutional Investor country risk ratings were first compiled in 1979, and are published now regularly, in March and September of every year, for an increasing number of countries, which reached 145 in 2000. The Institutional Investor ratings are numerical, ranging from 0 to 100, with 100 corresponding to the lowest chance of default. The Institutional Investor relies on evaluations of the creditworthiness of the countries to be rated, provided by economists and international banks, each respondent using their own criteria. Responses are aggregated by The Institutional Investor, greater weights being given to responses from institutions with higher worldwide exposure.

Correlation of difference matrices

Let us associate with any n-vector v its *[n x n]* "difference matrix" D whose component *(i,j)* is the difference of the *i's* and *j's* components of v. In Section 3.4.1, we have used that the correlation between any two n-vectors equals the correlation between their difference matrices. In spite of the elementary nature of this statement, we could not find it in the literature, and shall therefore provide here a formal proof of it.

Let us consider two n-dimensional vectors a and b, and two *[n x n]* dimensional difference matrices C and D, the elements of which c_{ij} and d_{ij} are given by:

$$\begin{cases} c_{ij} = a_i - a_j, \\ d_{ij} = b_i - b_j, \end{cases} \qquad i, j = 1,...,n \tag{14}$$

It can be seen from (14) that:

- C and D are anti-symmetric:

$$c_{ij} = -c_{ji} \text{ and } d_{ij} = -c_{ji}\ , \quad i, j = 1,...,n, \tag{15}$$

- C and D have diagonal elements equal to 0:

$$c_{ii} = d_{ii} = 0, \quad i = 1,...,n \tag{16}$$

- the average of the elements of C and D is equal to 0:

$$\overline{c} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} / n^2 = 0 \ \ , \ \overline{d} = \sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij} / n^2 = 0, \tag{17}$$

We shall show that the correlation between the vectors a and b,

$$\rho(a,b) = \frac{\frac{1}{n}\sum_{i=1}^{n}(a_i - \overline{a})(b_i - \overline{b})}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(a_i - \overline{a})^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n}(b_i - \overline{b})^2}} \tag{18}$$

is equal to the correlation between the matrices C and D

$$\rho(C,D) = \frac{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(c_{ij} - \overline{c})(d_{ij} - \overline{d})}{\sqrt{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(c_{ij} - \overline{c})^2}\sqrt{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(d_{ij} - \overline{d})^2}} = \frac{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)(b_i - b_j)}{\sqrt{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)^2}\sqrt{\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(b_i - b_j)^2}} \tag{19}$$

The expression $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)^2$ can be rewritten as

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}[(a_i - a_j)^2 + (a_j - a_i)^2] = 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i - a_j)^2 \tag{20}$$

Also, the expression $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)(b_i - b_j)$ can be rewritten as

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}[(a_i - a_j)(b_i - b_j) + (a_j - a_i)(b_j - b_i)] = 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i - a_j)(b_i - b_j) \tag{21}$$

Using (20) and (21), the correlation between the matrices C and D (10) can be rewritten as :

$$\rho(C,D)=\frac{\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i-a_j)(b_i-b_j)}{\sqrt{\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i-a_j)^2}\sqrt{\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(b_i-b_j)^2}} \qquad (22)$$

Similarly, the expression $\frac{1}{n}\sum_{i=1}^{n}(a_i-\overline{a})^2$ in (18) can be rewritten as

$$\frac{1}{n}\sum_{i=1}^{n}(a_i-\frac{1}{n}\sum_{j=1}^{n}a_j)(a_i-\overline{a})=\frac{1}{n^2}\sum_{i=1}^{n}(\sum_{j=1}^{n}(a_i-a_j))(a_i-\overline{a})=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i-a_j)(a_i-\overline{a})$$
$$=\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}[(a_i-a_j)(a_i-\overline{a})+(a_j-a_i)(a_j-\overline{a})]=\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i-a_j)(a_i-\overline{a}-a_j+\overline{a})=\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i-a_j)^2 \quad (23),$$

while the expression $\frac{1}{n}\sum_{i=1}^{n}(a_i-\overline{a})(b_i-\overline{b})$ in (18) can be rewritten as

$$\frac{1}{n}\sum_{i=1}^{n}(a_i-\frac{1}{n}\sum_{j=1}^{n}a_j)(b_i-\overline{b})=\frac{1}{n^2}\sum_{i=1}^{n}(\sum_{j=1}^{n}(a_i-a_j))(b_i-\overline{b})=\frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i-a_j)(b_i-\overline{b})$$
$$=\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}[(a_i-a_j)(b_i-\overline{b})+(a_j-a_i)(b_j-\overline{b})]=\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i-a_j)(b_i-\overline{b}-b_j+\overline{b}) \qquad (24)$$
$$=\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i-a_j)(b_i-b_j)$$

Using (23) and (24), the correlation between the vectors a and b (19) can be rewritten as

$$\rho(a,b)=\frac{\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i-a_j)(b_i-b_j)}{\sqrt{\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_i-a_j)^2}\sqrt{\frac{1}{n^2}\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(b_i-b_j)^2}}, \qquad (25)$$

showing its equality to $\rho(C,D)$. QED.