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Abstract: Recent advances in computation technology for decision/simulation and uncertainty analyses have revived interest in the the triangular distribution and its use to describe uncertainty of bounded input phenomena. The trapezoidal distribution, explicitly suggested by Pouliquen (1970) in the framework of risk and uncertainty analysis, is a generalization of the triangular distribution that allows for the specification of the modal value by means of a range of values rather than a single point estimate. While the trapezoidal and the triangular distributions are restricted to linear geometric forms in the successive stages of the distribution, the generalized trapezoidal (GT) distribution introduced by van Dorp and Kotz (2003) allows for a non-linear behavior at its tails and a linear incline (or decline) in the central stage. In this paper we shall develop two novel elicitation procedures for the parameters of a special case of the GT family by restricting ourselves to a uniform (horizontal) central stage in accordance with the central stage of the original trapezoidal distribution. We shall demonstrate the potential effect of allowing elicitation of the whole modal range, rather than the narrower point estimate, in an illustrative Monte Carlo analysis for a small decision tree.

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1. Introduction

In a recent survey paper a leading Bayesian statistician, O'Hagan (2006), explicitly mentions a need for advances in elicitation techniques for prior distributions in Bayesian Analyses, but also acknowledges the importance of their development for those areas where the elicited distribution can not be combined with evidence from data, because the expert opinion is essentially all the available knowledge. Garthwaite, Kadana and O'Hagan (2005) provide a comprehensive review on the topic of eliciting probability distributions dealing with a wide variety of topics, such as ,e.g., the elicitation process, heuristics and biases, fitting distributions to an expert's summaries, expert calibration and group elicitation methods. We encourage the reader to review the bibliography of Garthwaite, Kadana and O'Hagan (2005) which is impressive and contains over 100 references.

The topic of this paper deals with fitting a specific parametric distribution to a set of summaries elicited from an expert. Experts are traditionally classified into two, usually unrelated, groups: 1) *substantive* experts (also known as technical experts or domain experts) who are knowledgeable about the subject matter at hand and 2) *normative* experts mainly possessing knowledge of the appropriate quantitative analysis techniques (see, e.g., De Wispelare *et al.* (1995) and Pulkkinen and Simola (2000)). In the absence of data and in the context of decision/simulation and uncertainty analyses, substantive experts are used (primarily by necessity) to specify input distributions.

Advances in decision/simulation and uncertainty analysis methodology and their penetration into applied sciences and engineering during the last several decades (recall − by now standard − tools such as Decision Tool Suite by the Palisade Corporation, Crystal Ball by Decision Engineering, and ARENA by Rockwell Software) have reinvigorated the use of distributions with bounded support (that were not initially popular options). Integration of graphically interactive and statistical procedures for bounded input distribution modeling has become a topic of research (see, e.g., DeBrota *et al.* (1989), AbouRizk *et al.* (1992) and Wagner and Wilson (1995, 1996)) in order to facilitate their elicitation by experts. AbouRizk *et al.* (1992) have developed software with a graphical user interface (GUI) to ease fitting of beta distributions using a variety of methods and DeBrota *et al.* (1989) have developed software for fitting bounded Johnson S_B distributions. Wagner and Wilson

(1995, 1996) introduced univariate Bézier distributions (or curves), which are a variant of spline functions, and the software tool PRIME with a GUI to specify them. All these methods involve the *requirement* of specifying the lower and upper bounds of the distribution's support. While the system of Bézier distributions allows for great flexibility in input distribution modeling for stochastic simulations, Wagner and Wilson (1996) point out that random variate generation from a Bézier distribution is at present computationally inefficient since its F^{-1} (the inverse cumulative distribution function (cdf)) cannot be expressed in a closed form. The same applies for the beta or Johnson S_B distributions. Fortunately, triangular, trapezoidal and generalized trapezoidal distributions do have closed form cdf's.

Trapezoidal distributions have been advocated for use in risk analysis problems, initially by Pouliquen (1970) and more recently by Herrerías and Calvete (1987), Herrerías (1989), Powell and Wilson (1997) and Garvey (2000). Other applications of trapezoidal distributions are prominent in applied physics problems (see, e.g., Davis and Sorenson (1969), Nakao and Iwaki (2000), Sentenac *et al.* (2000)) and medical ones, specifically in the screening and detection of cancer (see, e.g., Flehinger and Kimmel (1987), Brown (1999) and Kimmel and Gorlova (2003)). Trapezoidal distributions have also been used as membership functions in fuzzy set theory (see, e.g., Chen and Hwang (1992) and Bardosi and Fodor (2004)).

Figure 1A plots a trapezoidal probability density function (pdf) suggested by Pouliquen (1970) with the boundary parameters $a = 0,\ b = 0.3,\ c = 0.5$ and $d = 1.$ The trapezoidal pdf depicted in Figure 1A is a generalization of the "classical" triangular distribution dating back as far as Simpson[5] (1755, 1757). Analogously to the triangular distribution, the trapezoidal distribution is appealing in practice mainly due to the ease of the physical interpretation of its parameters a, b, c and d. This would allow for their straightforward elicitation via a substantive expert knowledgeable about an uncertain phenomenon represented by the distribution. However, the requirement of specifying the bounds when using input distributions with bounded support in decision/simulation and uncertainty

[5]Thomas Simpson (1710-1761) a prolific writer of mathematical textbooks and able teacher at the Royal Military Academy in Wolwich (England) made original and important contributions to statistics and actuarial sciences.

analyses poses some challenges. Although the use of bounded distributions in the absence of data is by now prevalent, the fact that the lower and upper bounds of an uncertain phenomenon as a rule fall outside of the accumulated experience of a substantive expert (see, e.g., Selvidge (1980), Davidson and Cooper (1980), Alpert and Raiffa (1982), Keefer and Verdini (1993)) is rarely acknowledged. Instead these authors suggest the elicitation of lower and upper quantiles instead. Keefer and Verdini (1993) solved for the lower and upper bounds of a triangular distribution in the case when a point estimate for its mode is also available. Kotz and Van Dorp (2006) extended Keefer and Verdini's (1993) procedure for Two-Sided Power (TSP) distributions that are generalizations of triangular distribution allowing for non-linear behavior in the two tails.

Van Dorp and Kotz (2003) provide the probability density function (pdf) of the *Generalized* Trapezoidal (GT) distribution with parameters $\alpha, a, b, c, d,\ m$ and n given by

$$f_X(x|\Theta) = \mathcal{C}(\Theta) \times \begin{cases} \alpha\left(\frac{x-a}{b-a}\right)^{m-1}, & \text{for } a \leq x < b \\ (1-\alpha)(\frac{x-b}{c-b}) + \alpha, & \text{for } b \leq x < c \\ \left(\frac{d-x}{d-c}\right)^{n-1}, & \text{for } c \leq x < d \end{cases} \tag{1}$$

where the parameter vector $\Theta = \{a, b, c, d, m, n, \alpha\}$, $a < b < c < d$ and $m, n, \alpha > 0$ and the normalizing constant is

$$\mathcal{C}(\Theta) = \frac{2mn}{2\alpha(b-a)n + (\alpha+1)(c-b)mn + 2(d-c)m}. \tag{2}$$

The parameter $\alpha = f_X(b|\Theta)/f_X(c|\Theta)$ and is referred to as a boundary ratio parameter. The generalization (1) allows for non-linear behavior in the tails of the pdf via the tail parameters m and n and a linear incline (or decline) of the pdf in the central stage by setting the boundary ratio parameter $\alpha \neq 1$. It possesses a closed form cdf. Figure 1B plots a generalization of the pdf presented in Figure 1A with the same boundary parameters a, b, c and d and the additional parameter values $\alpha = 1\frac{1}{4},\ m = 3$ and $n = 5$. By substituting $n = m = 2$ and $\alpha = 1$ in (1) the pdf (1) reduces to the "classical trapezoidal" pdf. Apparently, elicitation procedures for the additional parameters of the GT distribution have not so far been developed. These type of procedures may be useful for its application in problems of decision/risk and uncertainty analysis.

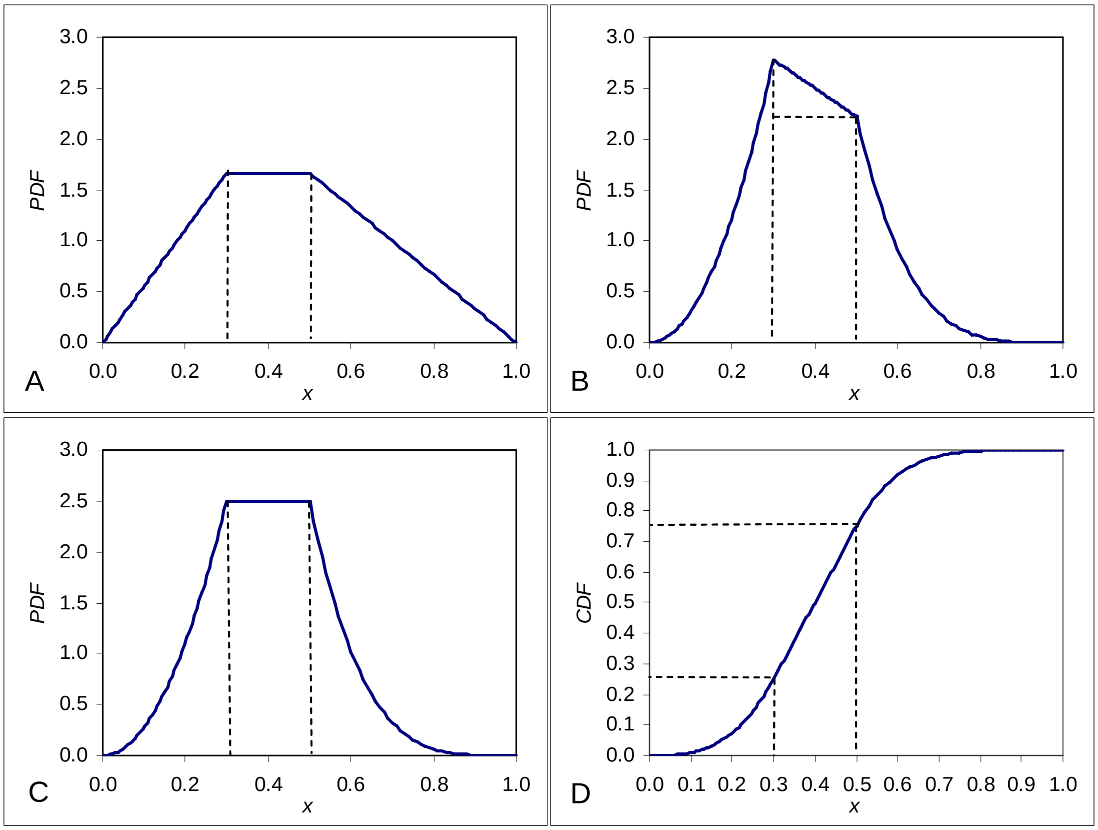

Fig. 1. Generalized trapezoidal (GT) densities with common boundary parameters $a = 0, b = 0.3, c = 0.5$ and $d = 1$; A: Original trapezoidal pdf with $m = 2,\ n = 2,\ \alpha = 1$, B: GT pdf with $m = 3,\ n = 5$, and $\alpha = 1\frac{1}{4}$, C: GT Uniform pdf with $m = 3,\ n = 5$, and $\alpha = 1$, D: GTU cdf with $m = 3,\ n = 5$, and $\alpha = 1$.

In the remainder of this paper we shall restrict ourselves to the analysis of GT distributions with a uniform central stage. This is achieved by setting $\alpha = 1$ in (1) and (2) and referring to it as *Generalized Trapezoidal Uniform* (GTU) distributions it has the pdf:

$$f_X(x|\Phi) = \mathcal{C}(\Phi) \times \begin{cases} \left(\frac{x-a}{b-a}\right)^{m-1}, & \text{for } a \le x < b, \\ 1, & \text{for } b \le x < c, \\ \left(\frac{d-x}{d-c}\right)^{n-1}, & \text{for } c \le x < d, \end{cases} \tag{3}$$

where the parameter vector $\Phi = \{a, b, c, d, m, n\}$, $a < b < c < d$ and $m, n > 0$ and the normalizing constant $\mathcal{C}(\Phi)$ is given by

$$\mathcal{C}(\Phi) = \frac{mn}{(b-a)n + (c-b)mn + (d-c)m}. \tag{4}$$

Figure 1C (Figure 1D) displays the GTU pdf (cdf) with $\alpha = 1$ in Figure 1B. Note that the modal value of the GTU pdf (3) is attained for all values in the central stage $[b, c]$. Hence, similarly to the original trapezoidal distribution (Figure 1A) one may directly elicit this modal range by means of a substantive expert (who may be more "comfortable" here, being relieved of providing a fixed point estimate for the modal value as required for a triangular distribution). Unfortunately, this is not plausible for the more general GT family (1) involving the boundary ratio parameter α (see, e.g., van Dorp and Kotz (2003)).

In the remainder of this paper we shall propose two elicitation procedures for the parameters of GTU distributions (3). In Section 2, the first method will be presented assuming that the boundary parameters a and d are known due to natural boundary constraints such as, for example, a return on investment (ROI) or a probability having a natural support $[0, 1]$. After eliciting the central stage bounds b and c, the tail parameters m and n will indirectly be elicited from a substantive expert following the fixed interval method mentioned in Garthwaite, Kadana and O'Hagan (2005). The second method, dealing with unknown boundary parameters a and d to be discussed in Section 3, elicits also a lower $a_p < b$ and upper quantiles $d_r > c$ which are used to solve for the lower a and the upper d bounds (p and r are usually assumed to be equal to 0.05 and 0.95 or 0.10 and 0.90, respectively). In Section 4, we present an illustrative decision tree example that compares the potential effect of eliciting the information in Section 3 with eliciting a point estimate for the most likely value m and the lower and upper quantiles $a_{0.10}$ and $d_{0.90}$.

2. Indirect elicitation of tail parameters with fixed lower and upper bounds

Mixing distributions is common practice in dealing with e.g. Phase-Type, Erlang, Poisson and Normal distributions (see, e.g., Johnson and Taaffe (1991) and Karlis and Xekalaki (1999)). The pdf

(3) may be expressed as a mixture (see, e.g., van Dorp and Kotz (2003)) involving three densities $f_{X_1}, f_{X_2}, f_{X_3}$ with bounded support, such that

$$f_X(x|\Phi) = \sum_{i=1}^{3} \pi_i f_{X_i}(x|\Phi), \quad \sum_{i=1}^{3} \pi_i = 1,\ \pi_i > 0, \tag{5}$$

where as above $\Phi = \{a, b, c, d, m, n\}$. Here

$$\begin{aligned} &f_{X_1}(x|\Phi) = f_{X_1}(x|a, b, m) = \left(\frac{m}{b-a}\right)\left(\frac{x-a}{b-a}\right)^{m-1}, \\ &a \le x < b,\ m > 0, \end{aligned} \tag{6}$$

$$f_{X_2}(x|\Phi) = f_{X_2}(x|b, c) = \frac{1}{c-b}, b \le x \le c, \tag{7}$$

and

$$\begin{aligned} &f_{X_3}(x|\Phi) = f_{X_3}(x|c, d, n) = \left(\frac{n}{d-c}\right)\left(\frac{d-x}{d-c}\right)^{n-1}, \\ &c \le x < d,\ n > 0. \end{aligned} \tag{8}$$

The mixture probabilities π_i, $i = 1, 2, 3$, are

$$\begin{cases} \pi_1 = \mathcal{C}(\Phi)(b-a)/m, \\ \pi_2 = \mathcal{C}(\Phi)(c-b), \\ \pi_3 = \mathcal{C}(\Phi)(d-c)/n, \end{cases} \tag{9}$$

where $\mathcal{C}(\Phi)$ is given by (4). Observe that the mixture weight of the first stage π_1 decreases as its tail parameter m increases. A similar observation can be made for the third stage with obvious modification.

After some algebraic manipulations we derive from $(5) - (9)$

$$E[X|\Phi] = \mathcal{C}(\Phi) \times \left[\frac{a+mb}{m(m+1)}(b-a) + \frac{b+c}{2}(c-b) + \frac{nc+d}{n(n+1)}(d-c)\right] \tag{10}$$

and

$$E[X^2|\Phi] = \mathcal{C}(\Phi) \times \Big\{ \Big[\frac{a^2}{m} + \frac{2a(b-a)}{m+1} + \frac{(b-a)^2}{m+2}\Big](b-a) + \\ \frac{b^2+bc+c^2}{3}(c-b) + \Big[\frac{d^2}{n} - \frac{2d(d-c)}{n+1} + \frac{(d-c)^2}{n+2}\Big](d-c)\Big]\Big\}. \tag{11}$$

Note that in expressions (10) and (11) the second factor is a weighted sum of the support widths $(b-a)$, $(c-b)$ and $(d-c)$, where the weights of the first and the third terms are also, but not solely, determined by the tail parameters m and n, respectively. Expressions (10) and (11) allow for straightforward evaluation of the variance $Var[X] = \ E[X^2|\Phi] - \ E^2[X|\Phi]$.

Finally, from $(5)\ -(9)$ one straightforwardly obtains the following convenient form for the cdf of (5) (equivalently (3))

$$F_X(x|\Phi) = \begin{cases} \pi_1\left(\frac{x-a}{b-a}\right)^m, & a \le x \le b, \\ \pi_1 + \pi_2\frac{x-b}{c-b}, & b < x \le c, \\ 1-\pi_3\left(\frac{d-x}{d-c}\right)^n, & c < x \le d, \end{cases} \tag{12}$$

and quantile function (or inverse cdf F^{-1}) :

$$F_X^{-1}(y|\Phi) = \begin{cases} a + \left(\frac{y}{\pi_1}\right)^{1/m}(b-a), & 0 \le y \le \pi_1, \\ b + \frac{y-\pi_1}{\pi_2}(c-b), & \pi_1 < y \le 1-\pi_3, \\ d - \left(\frac{1-y}{\pi_3}\right)^{1/n}(d-c), & 1-\pi_3 < y \le 1, \end{cases} \tag{13}$$

where the mixture probabilities π_1, π_2 and π_3 are given by (9).

Assume now that the parameters a and d are known and that the modal range $[b, c]$ has been directly elicited from a substantive expert. We shall now proceed using the fixed interval method mentioned in Garthwaite, Kadana and O'Hagan (2005). Namely, we suggest eliciting the relative likelihoods π_2/π_1 and π_2/π_3 (or their reciprocals) of the uncertain quantity at hand falling in the central stage $[b, c]$ relative to the tails $[a, b)$ and $(c, d]$, respectively. Next, we directly solve for the tail parameters m and n utilizing relationships:

$$\begin{cases} \frac{\pi_2}{\pi_1} = m\frac{c-b}{b-a}, \\ \frac{\pi_2}{\pi_3} = n\frac{c-b}{d-c}. \end{cases} \Leftrightarrow \begin{cases} m = \frac{\pi_2(b-a)}{\pi_1(c-b)}, \\ n = \frac{\pi_2(d-c)}{\pi_3(c-b)}, \end{cases} \tag{14}$$

which immediately follow from (9). Observe that expression (14) implies that the ratio of the central stage and first (third) stage probability equals m (n) times the ratio of the widths of their corresponding supports.

Figure 1C (Figure 1D) depicts a GTU pdf (cdf) of a ROI (with the natural support $[0, 1]$) for which it is twice as likely for the uncertain quantity to fall in the modal range $[0.3, 0.5]$ as compared to the tails $[0, 0.3)$ and $(0.5, 1]$. Hence, $\pi_2/\pi_1 = \pi_2/\pi_3 = 2$. The parameter values $m = 3$ and $n = 5$ now follow directly from (12). Observe in Figure 1D that the quantiles $x_{0.25} = 0.3$ and $x_{0.75} = 0.5$.

3. Indirect elicitation of tail parameters and lower and upper bounds

The elicitation of lower and upper quantiles for a bounded uncertain quantity adheres to the prevailing view that lower and upper bounds of an uncertain phenomenon as a rule fall outside of the accumulated experience of a substantive expert (see, e.g., Selvidge (1980), Davidson and Cooper (1980), Alpert and Raiffa (1982), Keefer and Verdini (1993)). Forcing a substantive expert in such a scenario to provide strict lower and upper bound estimates may lead to a misrepresentation of uncertainty. The elicitation of the quantiles $a_{0.10}$ and $d_{0.90}$ was suggested by Keefer and Verdini (1993). Instead of specifying the values $p = 0.10$ and $r = 0.90$ an alternative procedure could be to request the substantive expert to specify some other quantile levels p and r that he/she is comfortable with. Thus we shall assume here that the lower and upper bound parameters a and d and tail parameters m and n are <u>unknown</u>.

Moreover, we shall assume that the bound parameters a, d and tail parameters m and n need to be determined from (i) a directly elicited modal range $[b, c]$, (ii) the relative likelihoods π_2/π_1 and π_2/π_3, and (iii) a lower $a_p < b$ and upper $d_r > c$ quantiles. The ratio π_2/π_1 (π_2/π_3) may be elicited here by eliciting the likelihood of the central stage $[b, c]$ relative to the uncertain quantity being less (larger) than the lower bound b (upper bound c) of this central stage. The probabilities associated with each interval are then obtained utilizing that their sum must be 1 (see, Garthwaite, Kadana and O'Hagan (2005)). From expression (14), we can immediately write the lower a and the upper d bounds as linear functions of the tails parameters m and n, respectively. Specifically,

$$\begin{cases} a = b - \frac{\pi_1(c-b)}{\pi_2} m \equiv a^*(m), \\ d = c + \frac{\pi_3(c-b)}{\pi_2} n \equiv d^*(n). \end{cases} \tag{15}$$

(Notation $a^*(m)$ $[d^*(n)]$ for a $[d]$ emphasizes their dependence on m $[n]$.) Note that the expressions (15) do not involve the lower a_p and upper d_r quantiles, but result directly from the relations (14) linking the probability in each stage of the GTU distribution with the width of the support of each stage via the tail parameters m and n.

In the Subsections 3.1 and 3.2 we shall derive two additional functions such that $a \equiv \widetilde{a}(m)$ and $d \equiv \widetilde{d}(n)$ that describe a non-linear relationship between the tails parameters m and n and the lower a and upper d bounds. These relationships do involve a_p and d_r and, under certain uniqueness conditions, we may solve for m (n) by setting $a^*(m) = \widetilde{a}(m)$ (by setting $d^*(n) = \widetilde{d}(n)$).

3.1. Solving for the left tail parameter m and the lower bound a

From the expressions for the GTU cdf (12) and the definition of a lower quantile $a_p < b$ we obtain (by substitution) that

$$F_X(a_p|\Phi) = \pi_1\Big(\frac{a_p - a}{b - a}\Big)^m = p \Leftrightarrow a = a_p - \frac{\lambda(m, p, \pi_1)}{1 - \lambda(m, p, \pi_1)}(b - a_p) \equiv \widetilde{a}(m), \tag{16}$$

where $p < \pi_1$ and

$$0 < \lambda(m, p, \pi_1) = (p/\pi_1)^{1/m} < 1. \tag{17}$$

Note that the left hand side (LHS) of expression (16) links the first stage probability π_1 to the quantile level p, the width of the support of the first stage $(b - a)$ and the distance $(a_p - a)$ from the lower bound to the lower quantile a_p.

Note that both $\widetilde{a}(m)$ and $a^*(m)$ are just expressions for the same parameter a in different situations. We may now solve for a tail parameter m satisfying the lower quantile constraint (14) by setting

$$\widetilde{a}(m) = a^*(m), \tag{18}$$

where $a^*(m)$ is the linear function defined by (15). In the appendix we shall prove that the LHS lower bound function $\widetilde{a}(m)$ in (18) is concave, strictly decreasing with the asymptote

$$\mathcal{A}(m) = \frac{b - a_p}{\log(\frac{p}{\pi_1})} m + \frac{a_p + b}{2}. \tag{19}$$

From $\widetilde{a}(m) < a_p$ for all $m > 0$ (see (16)) and the properties of the function $\widetilde{a}(m)$, it immediately follows that the number of solutions of equation (18) equals that of the equation

$$\mathcal{A}(m) = a^*(m). \tag{20}$$

However, this number can be at most one, since both functions in (20) are linear. From,

$$\frac{a_p + b}{2} = \mathcal{A}(0) < a^*(0) = b. \tag{21}$$

it next follows that a unique solution for the tail parameter m exists for equation (18) iff when the slope of the asymptote $\mathcal{A}(m)$ is less steep than that of the linear function $a^*(m)$ defined by (15), i.e.

$$\frac{b - a_p}{\log\left(\frac{p}{\pi_1}\right)} < -\frac{\pi_1(c - b)}{\pi_2} \Leftrightarrow b - \xi(c - b) < a_p, \tag{22}$$

where

$$\xi = \frac{\pi_1}{\pi_2} \log\left(\frac{\pi_1}{p}\right) > 0. \tag{23}$$

Hence, condition (22) determines both the existence and uniqueness of a solution for equation (18).

Figure 2A depicts both functions $a^*(m), \widetilde{a}(m)$ and the asymptote $\mathcal{A}(m)$ for the case that $b = 80, c = 100$, $\pi_2/\pi_1 = 1.5$ and $\pi_2/\pi_3 = 1$ and lower quantile $a_{0.10} = 71$. We have from $\pi_2/\pi_1 = 1.5$ and $\pi_2/\pi_3 = 1$ that

$$\pi_1 = \frac{1}{4},\ \pi_2 = \pi_3 = \frac{3}{8}. \tag{24}$$

For the data in Figure 2A the condition (22) reduces to

$$80 - 13\frac{1}{3}\log(2.5) \approx 67.783 < a_{0.10} = 71. \tag{25}$$

Hence, a unique solution of equation (18) exists for the data in Figure 2A. Solving for m using a standard root finding algorithm yields

$$m = 1.419 \tag{26}$$

and substituting $m = 1.419$ in either $\widetilde{a}(m)$ (16) or $a^*(m)$ (15) yields the lower bound

$$a = \widetilde{a}(m) = a^*(m) \approx 61.085. \tag{27}$$

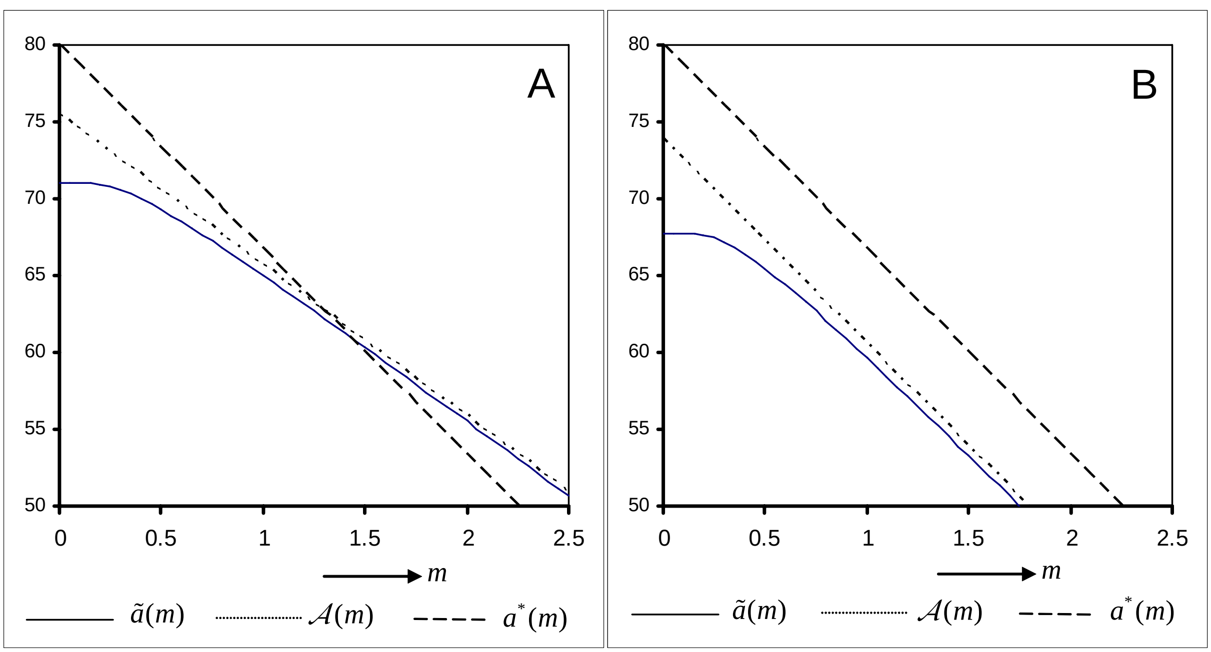

Fig. 2. Lower bound functions $a^*(m)$ (15) and $\widetilde{a}(m)$ (16) with its asymptote $\mathcal{A}(m)$ (19) for the data $b = 80, c = 100$, $\pi_1 = 25\%$, $\pi_2 = 37.5\%$; A: lower quantile $a_{0.10} = 71$, B: lower quantile $a_{0.10} \approx 67.783$.

Figure 2B plots the functions $\widetilde{a}(m)$, $a^*(m)$ and the asymptote $\mathcal{A}(m)$ for the boundary case (22) $a_{0.10} \approx 67.783$. Observe that in this case no solution for equations (18) and (20) exists.

From (22) and (23) we conclude that the pre-assigned quantile level p provides a lower threshold for the quantile a_p defined (22). This threshold is just a function of the width of the central stage $(c-b)$, its probability π_2, the first stage probability π_1 and the quantile level p. In case a substantive expert specifies a set of values for a_p, π_2, π_1 and b and c for which the condition (22) is not met, he/she may be given the option to revise his/her assessments utilizing the threshold value $b - \xi(c-b)$ in (22) as feedback. The use of feedback to enhance consistency in an expert's judgement is quite common (see, e.g., Denham and Mengersen (2007)).

3.2. Solving for the right tail parameter n and the upper bound d

After fully digesting the derivations in subsection 3.1 this subsection is straightforward. From the expression for the GTU cdf (12) and the definition of the upper quantile $d_r > c$ we obtain that

$$F_X(d_r|\Phi) = 1 - \pi_3\Big(\frac{d-x}{d-c}\Big)^n = r \Leftrightarrow d = d_r + \frac{\mu(n,r,\pi_3)}{1-\mu(n,r,\pi_3)}(d_r - c) \equiv \widetilde{d}(n), \qquad (28)$$

where $\pi_3 > 1 - r$ and

$$0 < \mu(n,r,\pi_3) = \{(1-r)/\pi_3\}^{1/n} < 1. \qquad (29)$$

Analogously as in Subsection 3.1, we may solve for the tail parameter n satisfying the upper quantile constraint (28) by setting

$$\widetilde{d}(n) = d^*(n), \qquad (30)$$

where $d^*(n)$ is the linear function defined by (15). The following properties of $\widetilde{d}(n)$ are derived in the appendix: $\widetilde{d}(0) = d_r$, $\widetilde{d}(n)$ is strictly increasing, is convex and possesses the asymptote

$$\mathcal{D}(n) = \frac{c - d_r}{\log(\frac{1-r}{\pi_3})}n + \frac{c+d_r}{2}. \qquad (32)$$

A unique solution to equation (30) exists iff

$$d_r < c + \psi(c - b), \tag{33}$$

where

$$\psi = \frac{\pi_3}{\pi_2}\log\left(\frac{\pi_3}{1-r}\right) > 0. \tag{34}$$

Figure 3A depicts both functions $\tilde{d}(n)$ defined by (28), $d^*(n)$ defined by (15) and the asymptote $\mathcal{D}(n)$ defined by (32) for the case that $b = 80, c = 100$, $\pi_2/\pi_1 = 1.5$ and $\pi_2/\pi_3 = 1$ and upper quantile $d_{0.90} = 121$. For this data a unique solution of (30) exists since the RHS threshold of (33) $c + \psi(c - b)$ equals

$$100 + 20\log(3.75) \approx 126.425 > d_{0.90} = 121. \tag{35}$$

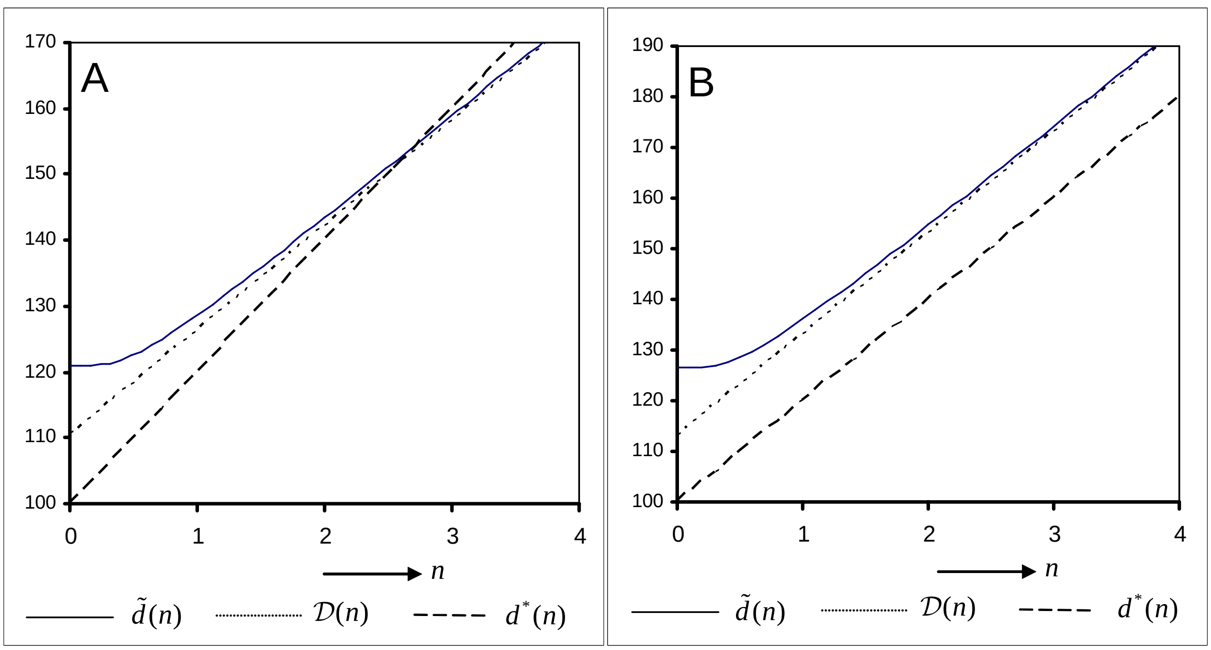

Fig. 3. Upper bound functions $d^*(n)$ (15) and $\tilde{d}(n)$ (28) with its asymptote $\mathcal{D}(n)$ for the data $b = 80, c = 100$, $\pi_2 = \pi_3 = 37.5\%$; A: upper quantile $d_{0.90} = 121$, B: upper quantile $d_{0.90} \approx 126.425$.

Solving for n using a standard root-finding algorithm yields

$$n = 2.757 \tag{36}$$

and substituting $n = 2.757$ in either $\widetilde{d}(n)$ defined by (28) or $d^*(n)$ defined by (15) yields the upper bound

$$d = \widetilde{d}(n) = d^*(m) \approx 155.135. \tag{37}$$

Figure 3B plots the functions $\widetilde{d}(m)$, $d^*(m)$ and the asymptote $\mathcal{D}(n)$ for the case of the LHS boundary $d_{0.90} \approx 155.135$ given by (37). Observe that here no solution exists for equation (30).

Finally, Figure 4 presents the GTU distribution which satisfies the constraints $b = 80$, $c = 100$, $\pi_2/\pi_1 = 1.5$ and $\pi_2/\pi_3 = 1$ and possesses lower $a_{0.10} = 71$ and upper $d_{0.90} = 121$ quantiles. Its mixture probabilities π_1, π_2 and π_3 (24) follow from $\pi_2/\pi_1 = 1.5$ and $\pi_2/\pi_3 = 1$. The unique tail parameters m and n that follow are provided by (26) and (36). The unique lower and upper bounds $a \approx 61.085$ and $d \approx 155.135$ are given by (27) and (37).

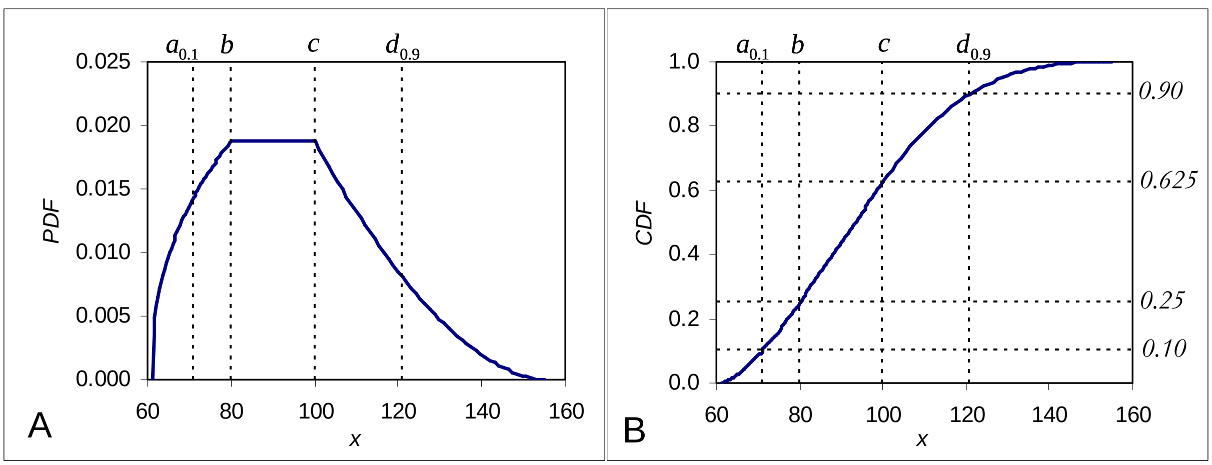

Fig. 4. GTU distribution (3) with $b = 80, c = 100$, $\pi_2/\pi_1 = 1.5,\ \pi_2/\pi_3 = 1,$ lower quantile $a_{0.10} = 71$ and upper quantile $d_{0.90} = 121$. The tail parameter values $m \approx 1.419,\ n \approx 2.757$, and the lower $a \approx 61.085$ and upper bounds $d \approx 155.135$ were determined utilizing the equations (16) and (27).

4. An illustrative decision tree example

It is not uncommon that in case of complicated uncertainty models in a decision tree or an influence diagram that a Monte Carlo analysis provides a convenient approach for solving the decision problem at hand (see, e.g., Clemen and Reilly (2001)). Consider the decision tree in Fig.5, which is a modified version of the scenario in Clemen and Reilly (2001), p. 461.

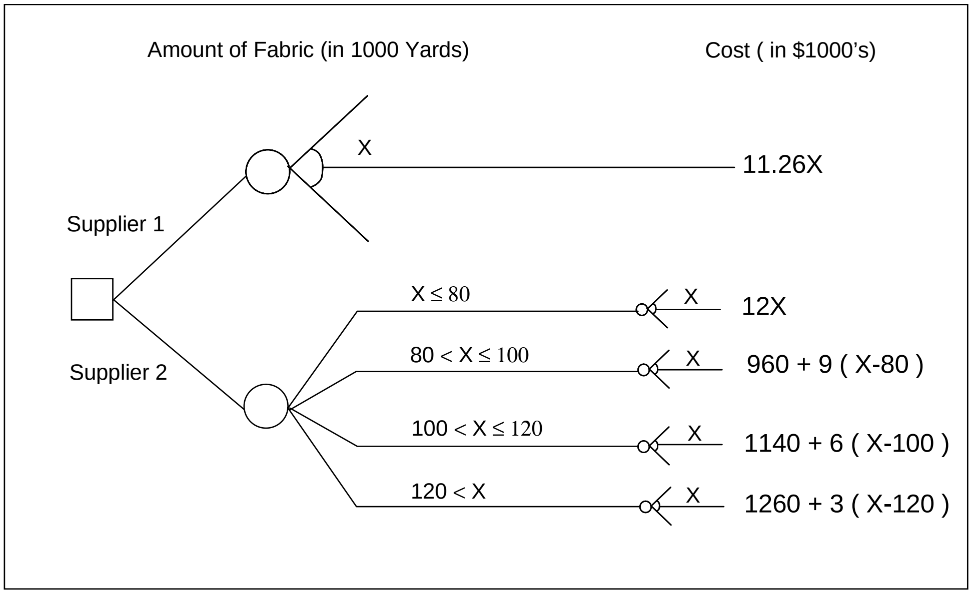

Fig. 5. Decision tree for the fabric buyer Janet Dawes

Here, the decision maker Janet Dawes (J.D.) is a purchaser for a factory that produces clothes who needs to choose between two suppliers of the fabric to produces a new lines of clothes for the upcoming season. Supplier 1 will supply the fabric at a cost of $11.26 regardless of the amount of fabric that J.D. orders. Supplier 2 has a more gradual pricing structure. The first 80,000 yards will cost $12.00 per yard. The next 20,000 will cost $9.00 per yard followed by a subsequent price drop of another $3.00 dollars for the following 20,000. Finally, Supplier 2 charges only $3.00 per yard for anything more than 120,000 yards.

After scrutinizing the uncertainty for the sales of the new clothing line in the upcoming season, and drawing from her past experience, J.D. assesses that she is 90% sure that the amount of fabric needed over the season will be above 71,000 yards, but with the same certainty level will not exceed 121,000 yards. In addition, she believes that the most likely value ranges between 80,000 and 100,000 yards. Finally, she assesses that it is 1.5 times more likely for the number of yards to fall within the estimated modal range 80,000 − 100,000 than being less than 80,000, while it is equally likely to be larger than 100,000 yards. (A somewhat optimistic assessment.)

A distribution that is consistent with J.D.'s degree of belief statements above is the GTU distribution (3) in Figure 4 $(a, b, c$ and d in \$1000 's). Using the Monte Carlo method we generate the cumulative risk profiles in Fig. 6 for both suppliers using a sample of size $10,000$ yielding an Expected Monetary Value (EMV) for supplier 1 of $\approx$ 1067.49 $\times$ \$1000 and an EMV of $\approx$ 1068.07 $\times$ \$1000 for the second. Hence, the difference between their EMV's is approximately \$584.76. Since J.D.'s objective is to minimize cost (and being devout risk neutral) she would choose Supplier 1 despite the larger standard deviation of 209.73 $\times$ \$1000 for Supplier 1 as compared to the 147.56 $\times$ \$1000 for Supplier 2.

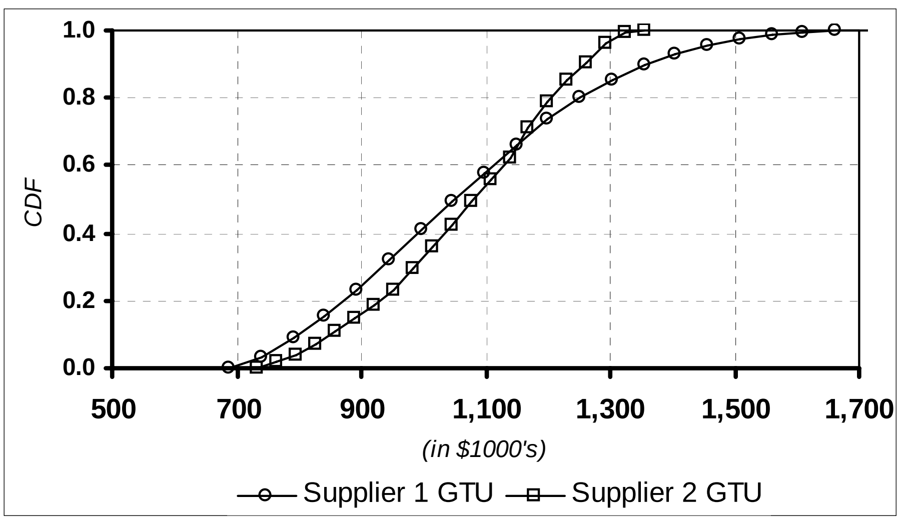

Fig. 6. Cumulative risk profiles for Supplier 1 and Supplier 2 for the decision tree in Fig.5 developed using 10,000 samples generated using the Monte Carlo method.

In the absence of a GTU distribution to match J.D.'s (substantive expert's) degree of belief, a traditional approach would have been for a normative expert (e.g. working for J.D.) to select say a triangular distribution that matches the lower quantile $x_{0.10} = 71,$ upper quantile $x_{0.90} = 121$ and carry out a sensitivity analysis by setting its most likely value to $m = 80$ and $m = 100.$ Lower and upper bounds may be obtained for these triangular distributions using e.g. the method of Keefer and Verdini (1993) or that of Kotz and van Dorp (2006). Another appropriate distribution for the normative expert's sensitivity analysis would have been a uniform distribution with the same 10% and 90% quantiles and having a mode at 80 or at 100 (albeit degenerates ones). In case of the uniform distribution this results in Supplier 2 having the lowest EMV $\approx 1079.80 \times \$1000$ amongst the two (unlike for the previous GTU analysis). The optimal decision also switches between Suppliers 1 and 2 with an optimal EMV of $\approx 1056.81 \times \$1000$ for Supplier 1 in case of the triangular distribution with the left mode 80 and an optimal EMV of $\approx 1083.38 \times \$1000$ for Supplier 2 in case of the triangular distribution with the right mode 100 (both having the same the lower $x_{0.10} = 71$ and upper quantiles $x_{0.90} = 121$).

Noting this switching and the differences in the values of the optimal EMV's, J.D.'s normative expert decides to plot the cumulative risk profiles for both Suppliers under these three scenarios. Figure 7 reveals that the differences between the cumulative risk profiles can be quite substantial.

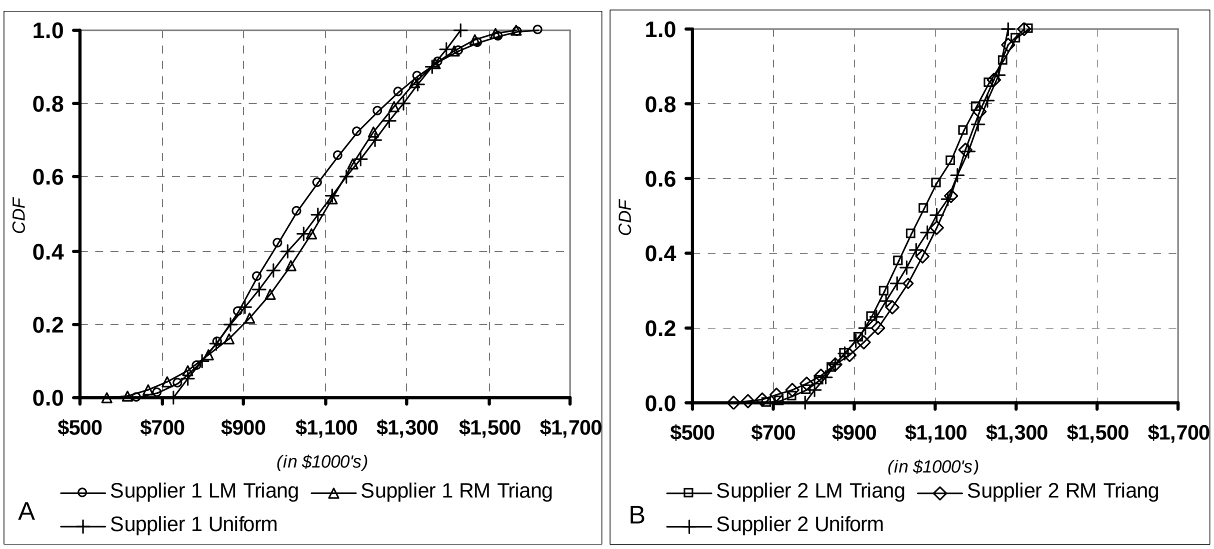

Fig. 7. Sensitivity analysis of cumulative risk profiles for Supplier 1 and Supplier 2 for the decision tree in Figure 5 developed using 10,000 samples generated using the Monte Carlo method. The distribution of the demand X was modeled using the Triangular Left Mode, Triangular Right Mode and Uniform distributions with the parameters specified in Table 1.

Observe a difference of at least 0.10 (at $1100 \times \$1000$) for the cumulative risk profiles of both suppliers, being solely a function of the choice of the distribution for the fabric yards required over the season and the particular mode being selected in the range from 80 to 100. This could pose a dilemma for the normative expert regarding which distribution and which mode to utilize (in the absence of the GTU distribution). Puzzled by the observed differences in Figure 7, the normative expert decides to model the distribution for X (fabric yards required) by means of beta, asymmetric Laplace (AL) distributions and gamma type distributions with the same lower $x_{0.10} = 71$ and upper quantiles $x_{0.90} = 121$ while switching between modes $m = 80$ and $m = 100$.

In the AL case it seems to be convenient to use the so-called "quantiles-mode parameterization" of Kotz and van Dorp (2005) with pdf

$$f_X(x|a_p, m, b_r) = \begin{cases} q\mathcal{A}Exp\Big\{-\mathcal{A}(m-x)\Big\} & x \leq m \\ \{1-q\}\mathcal{B}Exp\Big\{-\mathcal{B}(x-m)\Big\} & x > m, \end{cases} \tag{38}$$

where the coefficients $\mathcal{A}$ and $\mathcal{B}$ are

$$\mathcal{A} = \frac{Log\Big\{\frac{q}{p}\Big\}}{m - a_p} \text{ and } \mathcal{B} = \frac{Log\Big\{\frac{1-q}{1-r}\Big\}}{b_r - m}, \tag{39}$$

respectively (don't confuse the constant $\mathcal{A}$ here with the asymptote $\mathcal{A}(m)$ in (19)); The mode probability q in (38) and (39) is the unique solution of the equation

$$\frac{q}{\delta}Log\Big\{\frac{q}{p}\Big\} = \frac{1-q}{1-\delta}Log\Big\{\frac{1-q}{1-r}\Big\}, \tag{40}$$

where $\delta = (m - a_p)/(b_r - a_p)$. For the gamma distribution with support $[\gamma, \infty)$ we shall use the standard parameterization

$$f_X(x|\gamma, \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)}(x-\gamma)^{\alpha-1}exp\{-(x-\gamma)/\beta\}. \tag{41}$$

A reflected gamma distribution with support $(-\infty, \gamma]$ (with a tail towards the left) is required to match the right mode $m = 100$ and the quantiles $x_{0.10} = 71$ and $x_{0.90} = 121$. The various parameters settings for these distributions are presented in Table 1. While the distributions in Table

1 are consistent with the lower and upper quantiles of J.D.'s degree of belief and the mode is located within the specified range $[80, 100]$, neither of these distributions match her indirect estimates for $\pi_1 = Pr(X \leq 80) = 0.25$ and $1 - \pi_3 = Pr(X \leq 100) = 0.625$. Table 1 also contains for comparison two columns of the values for these probabilities.

Table 1. Parameter settings of various distributions with lower $x_{0.10} = 71$ and upper $x_{0.90} = 121$ quantiles, and the mode of either 80 or 100. Two columns contain values for $Pr(X \leq 80)$ and $Pr(X \leq 100)$, where X represents the number of fabric yards required in J.D.'s decision problem in Figure 5.

Distribution	$mode = 80$	$Pr(X \leq 80)$	$Pr(X \leq 100)$
Uniform	$a = 64.75,\ b = 127.25$	0.244	0.564
Triangular	$a \approx 56.635,\ m = 80,\ b \approx 144.950$	0.265	0.648
Beta	$a \approx 61.897, b \approx 299.885,\ \alpha = 2,\ \beta \approx 13.146$	0.293	0.687
Asymmetric Laplace	$a_{0.10} = 71,\ m = 80,\ b_{0.90} = 121,\ q(\infty) \approx 0.287$	0.287	0.721
Gamma	$\alpha \approx 2.470,\ \beta \approx 13.205,\ \gamma \approx 60.600$	0.298	0.698
Distribution	$mode = 100$	$Pr(X \leq 80)$	$Pr(X \leq 100)$
Uniform	$a = 64.75,\ b = 127.25$	0.244	0.564
Triangular	$a \approx 49.679,\ m = 80,\ b \approx 140.011$	0.202	0.557
Beta	$a \approx 53.495, b \approx 133.426,\ \alpha = 2,\ \beta \approx 1.7187$	0.214	0.553
Asymmetric Laplace	$a_{0.10} = 71,\ m = 100,\ b_{0.90} = 121,\ q(\infty) \approx 0.550$	0.172	0.550
Reflected Gamma	$\alpha \approx 35.376,\ \beta \approx 3.293,\ \gamma \approx 213.23$	0.192	0.545

Table 2 contains the values of the EMV and the standard deviations for both suppliers for the various choices of distributions presented in Table 1. We observe from Table 2 that the decision switches from Supplier 1 to Supplier 2 when one switches from the left mode 80 to the right one 100, while the standard deviations are always smaller for the Supplier 2 distributions, independently of the selected mode. To complete the sensitivity analysis, Figure 8 plots the cumulative risk profiles for both suppliers for the distributions in Table 1 grouping them by suppliers and the two modes.

Table 2. EMV and standard deviations of the cost distributions for the two suppliers using the GTU distributions in Figure 8 and the various distributions in Table 1, to model X, where represents the number of fabric yards required in J.D.'s decision problem in Figure 5.

	EMV Supplier 1 (in \$1000's)	EMV Supplier 2 in \$1000's)	EMV Supplier 1 < EMV Supplier 2? False = 0, True = 1	St. Dev. MV Supplier 1 (in \$1000's)	St. Dev. MV Supplier 2 (in \$1000's)	St. Dev. Supplier 1 < St. Dev. Supplier 2? False = 0, True = 1
GTU	1,067.49	1,068.07	1	209.73	147.56	0
Uniform	1,081.29	1,079.80	0	202.17	147.59	0
Triangular LM	1,056.81	1,060.23	1	209.36	149.03	0
Triangular RM	1,087.53	1,083.38	0	207.22	152.82	0
Beta LM	1,050.55	1,052.74	1	224.50	148.47	0
Beta RM	1,086.68	1,082.98	0	205.69	151.76	0
Gamma LM	1,049.15	1,050.50	1	232.13	149.66	0
Gamma RM	1,089.02	1,082.31	0	220.13	166.74	0
Laplace LM	1,040.67	1,042.00	1	245.76	160.26	0
Laplace RM	1,091.34	1,081.30	0	240.26	182.54	0

Observe from Figure 8 that, despite the equality of a mode and a lower and upper quantiles amongst the distributions for X, the differences between the cumulative risk profiles (CRP's) for the suppliers can be quite substantial. A largest difference of approximately 0.18 at $1100 \times \$1000$ for both suppliers in Figures 8A and 8B certainly decreases the comfort level of J.D.'s normative expert even more. Apparently, a lower and upper quantiles estimate and a single mode do not provide sufficient amount of information in order to fit and choose a specific distribution. Kotz and Van Dorp (2006) arrived at the same conclusion using a different scenario and suggest the elicitation of one additional quantile to fit uniquely a TSP distribution with density

$$f_X(x|a,m,b,n) = \begin{cases} \frac{n}{(b-a)}\left(\frac{x-a}{m-a}\right)^{n-1} & a < x \leq m \\ \frac{n}{(b-a)}\left(\frac{b-x}{b-m}\right)^{n-1} & m \leq x < b, \end{cases} \tag{42}$$

where $n > 0,$ to determine the power parameter n. (This TSP distribution becomes a triangular distribution for $n = 2$).

One approach that the decision maker J.D could follow here is to attempt to fit her degree of belief statements $Pr(X \leq 80) = 0.25$ and $Pr(X \leq 100) = 0.625$ amongst the distribution in Table 1. The distribution that matches most closely (but not exactly) is the triangular distribution with a mode at 80 yielding the same optimal decision for Supplier 1 as obtained in the earlier analysis using the GTU distribution in Fig. 6, which does accurately matche these statements.

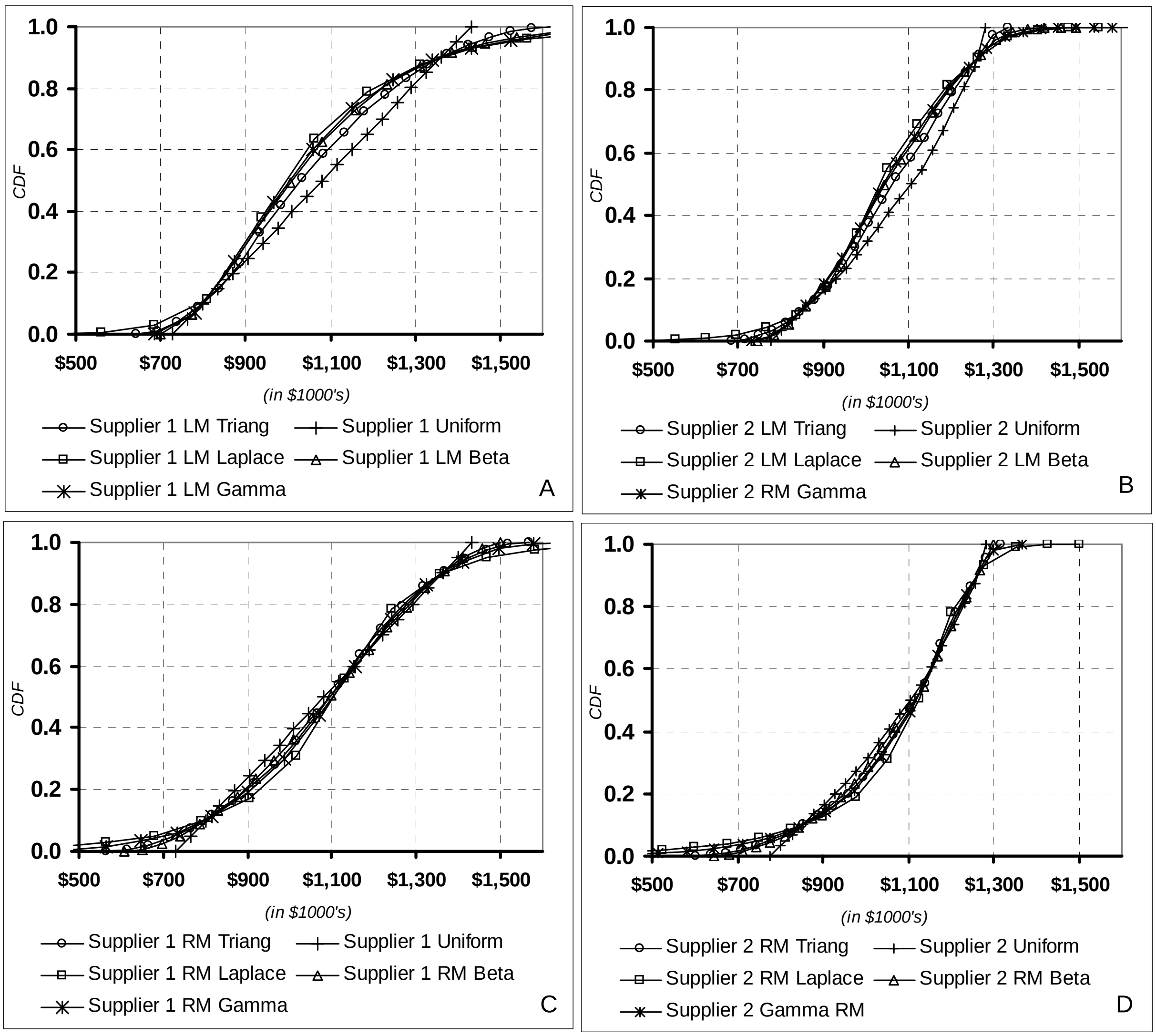

Fig. 8. Sensitivity analysis of cumulative risk profiles for suppliers 1 and 2 for the decision tree in Figure 5 developed by means of 10,000 samples generated using the Monte Carlo method. The distributions of the demand X were modeled using 9 distributions in Table 1 which share $x_{0.10} = 71,\ x_{0.90} = 121$.

A: Supplier 1 CRP's when the distribution of X has the mode at 80; B: Supplier 2 CRP's when the distribution of X has the mode at 80; C: Supplier 1 CRP's when the distribution of X has the mode at 100; D: Supplier 2 CRP's when the distribution of X has the mode at 100;

Moreover, the GTU distribution does match J.D.'s modal range statement of $[80, 100]$ while the triangular distribution does not. We are quite at ease that a normative expert would be most comfortable with the GTU distribution when accepting J.D.'s degree of belief statements.

Concluding Remarks

The scenario in this paper has been specifically constructed with the displayed behavior in mind. It provides an illustration of what could be the consequences when a substantive expert specifies a modal range rather than a point estimate (in the absence of the generalized trapezoidal-uniform distribution). We have succeeded to construct this example utilizing the fact that the specification of lower and upper quantiles and a single mode prevent to fit a univariate continuous uniquely, and the differences could be quite substantial. Hence, additional information ought to be elicited to further refine such a fitting procedure. In case of the generalized trapezoidal-uniform distribution distribution, this information is elicited in terms of the relative likilihood of the first and third stages of the distribution as compared to its central modal range. The purpose of introducing the generalized trapezoidal-uniform distribution distribution here is not to replace well established distributions such as the beta, triangular, asymmetric Laplace, gamma or any other appropriate distribution. It, however, does provide one additional distribution to the arsenal of available ditribution to match to a substantive experts degree of beliefs statements.

We have described two elicitation procedures for generalized trapezoidal-uniform distribution distributions defined in (3) and (4). The first method assumes that the lower and upper bounds are known based on some physical boundary constraints. The second one, which is possibly more realistic albeit more involved, solves for the lower and upper bounds by eliciting a lower and upper quantiles. We believe that these elicitation procedures will facilitate an application of GTU distributions in problems of decision, risk and uncertainty analysis.

Acknowledgement

We are indebted to Professor Samuel Kotz for his time to provide some references, comments and suggestions in the course of the development of this paper. This paper was developed with partial support from the Decision Sciences Institute at the George Washington University.

Appendix. Mathematical Properties of non-linear lower and upper bound functions involved in the elicitation procedure.

Here we shall show that: (i) the function

$$f(n) = \alpha - \frac{x^{1/n}}{1 - x^{1/n}}(\beta - \alpha), n \geq 0, \tag{A.1}$$

with the auxiliary parameters $x \in (0, 1)$, α and β is a strictly decreasing concave (increasing convex) function in n provided $\alpha < \beta$ $(\alpha > \beta)$. Also, (ii)

$$\lim_{n \to \infty} f(n) = \frac{\beta - \alpha}{\log(x)}n + \frac{\alpha + \beta}{2}. \tag{A.2}$$

and finally (iii) for all $n \geq 0$:

$$f(n) < \frac{\beta - \alpha}{\log(x)}n + \frac{\alpha + \beta}{2} \Leftrightarrow \alpha < \beta. \tag{A.3}$$

In other words, the RHS of $(A.2)$ (which also appears in $(A.3)$) is an asymptote of the function $f(n)$ defined by $(A.1)$ and the two curves do not intersect.

Lemma 1: *The function* $f(n)$ *defined by* $(A.1)$ *is strictly decreasing and concave for* $\alpha < \beta$ *and is strictly increasing and concave for* $\alpha > \beta$.

Proof: Assume, without loss of generality, that $\alpha < \beta$ or equivalently $\beta - \alpha > 0$. (Obvious modifications can be carried out for the case $\alpha > \beta$). Taking the first order derivative of $f(n)$ with respect to $n > 0$ we have

$$\frac{d}{dn}f(n) = (\beta - \alpha)\log(x)\ \frac{x^{1/n}}{n^2(1 - x^{1/n})^2} < 0 \tag{A.4}$$

since $x \in (0,1)$. Hence, the function $f(n)$ is strictly decreasing when $\beta - \alpha > 0$. Taking the second order derivative with respect to n we have

$$\frac{d^2}{dn^2} f(n) = -\frac{\log(x)}{n}\left[\frac{1}{n}\frac{1+x^{1/n}}{1-x^{1/n}} + \frac{2}{\log(x)}\right]\frac{d}{dn} f(n). \tag{A.5}$$

To prove that $f(n)$ is a concave function it is required to show that $\frac{d^2}{dn^2} f(n) < 0$ for all $n > 0$. Utilizing $(A.4)$ and $x \in (0,1)$ it is sufficient to prove that, for all $n > 0$,

$$\frac{1+x^{1/n}}{1-x^{1/n}} \times n^{-1} + \frac{2}{\log(x)} > 0 \Leftrightarrow g(n) > -\frac{2}{\log(x)}, \tag{A.6}$$

where

$$g(n) = \frac{1+x^{1/n}}{1-x^{1/n}} \times n^{-1},\ n > 0. \tag{A.7}$$

From $(A.7)$ and $x \in (0,1)$ it follows immediately that $g(n) \to \infty$ as $n \downarrow 0$ since $x^{1/n} \downarrow 0$ as $n \downarrow 0$. Letting now $n \to \infty$, after some algebraic manipulations involving $g(n)$ and applying the L' Hopital rule we easily obtain

$$\lim_{n \to \infty} g(n) = -\frac{2}{\log(x)}. \tag{A.8}$$

Thus to confirm the condition $(A.6)$ for all $n > 0$ it is only required to prove that $g(n)$ is a strictly decreasing function.

Now denoting $m \equiv x^{1/n} \in (0,1)$ (recall $x \in (0,1)$), we have for $n > 0$:

$$m = \sqrt[n]{x} \Leftrightarrow n = \frac{\log(x)}{\log(m)}, \tag{A.9}$$

$$\frac{dm}{dn} = -\frac{\log(x)}{n^2}\sqrt[n]{x} = -\frac{m\log^2(m)}{\log(x)}. \tag{A.10}$$

Taking the derivative of $(A.7)$ with respect to n and utilizing $(A.9)$ and $(A.10)$ we arrive at

$$\frac{d}{dn} g(n) = \frac{dm}{dn}\frac{d}{dm} g(m) = -\frac{\log^2(m)}{\{1-m\}^2\log^2(x)} \times h(m) \tag{A.11}$$

where

$$h(m) = 2m\log(m) + (1 - m^2). \qquad (A.12)$$

Since the multiplier of $h(m)$ in the RHS of $(A.11)$ is strictly negative it follows from $(A.11)$, $m \equiv x^{1/n}$ and $x \in (0,1)$ that

$$\frac{d}{dn}g(n) < 0 \text{ for all } n > 0 \Leftrightarrow h(m) > 0 \text{ for all } m \in (0,1). \qquad (A.13)$$

From the definition of $h(m)$ in $(A.12)$, $m \in (0,1)$, and noting that

$$h(0) = 1 \wedge h(1) = 0, \qquad (A.14)$$

and observing the form of a graph of $h(m)$ in Figure 9, we verify that indeed $h(m) > 0$ for all $m \in (0,1)$ (and thus from $(A.13)$ and $(A.9)$ it follows that $g(n)$ is a strictly decreasing function).☐

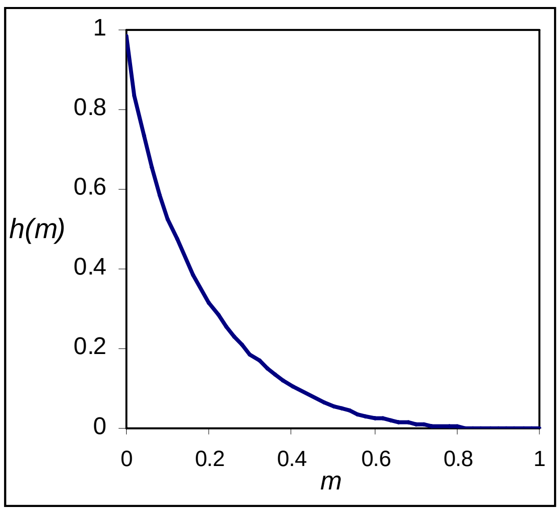

Fig. 9. A Plot of the function $h(m)$ for $m \in (0,1)$ defined by $(A.13)$.

Admittedly the plot in Figure 9 cannot serve as a formal proof. However, it is not difficult but somewhat tedious to show that the function $h(m)$ is strictly decreasing. This together with $(A.14)$ mathematically proves the RHS assertion of $(A.13)$[6]. We invite the readers to provide an alternative (possibly simpler) proof that the function $g(n)$ defined by $(A.7)$ is strictly decreasing for $n > 0$.

[6]A proof of $h(m)$ being strictly decreasing is available from the authors upon request.

Lemma 2: *The conditions* $(A.2)$ *and* $(A.3)$ *hold for the function* $f(n)$ *defined by* $(A.1)$.

Proof: From the derivative $\frac{d}{dn}f(n)$ given by $(A.4)$, substituting $m = x^{1/n}$, $x \in (0,1)$, and applying the L'Hopital rule twice we have

$$\lim_{n \to \infty} \frac{d}{dn} f(n) = \frac{\beta - \alpha}{\log(x)} \lim_{m \to 1} \frac{m\log^2(m)}{(1-m)^2} = \frac{\beta - \alpha}{\log(x)}. \qquad (A.15)$$

Thus

$$\lim_{n \to \infty} f(n) = \frac{\beta - \alpha}{\log(x)} n + \mathcal{C} \qquad (A.16)$$

where $\mathcal{C}$ is a constant. To show that the value of $\mathcal{C} = (\alpha + \beta)/2$ (see $(A.2)$) we evaluate

$$\lim_{n \to \infty} \left\{ f(n) - \frac{\beta - \alpha}{\log(x)} n \right\} = \lim_{n \to \infty} \frac{(\alpha - \beta x^{1/n})\log(x) - n(\beta - \alpha)(1 - x^{1/n})}{\log(x)(1 - x^{1/n})}. \qquad (A.17)$$

Substituting once more $m = x^{1/n}$ we have

$$\lim_{n \to \infty} \left\{ f(n) - \frac{\beta - \alpha}{\log(x)} n \right\} = \lim_{m \to 1} \frac{(\alpha - \beta m)\log(m) - (\beta - \alpha)(1 - m)}{\log(m)(1 - m)}. \qquad (A.18)$$

Applying of the L'Hopital rule twice to $(A.18)$ yields

$$\lim_{n \to \infty} \left\{ f(n) - \frac{\beta - \alpha}{\log(x)} n \right\} = \frac{\alpha + \beta}{2} \qquad (A.19)$$

and the assertion $(A.2)$ is valid. Condition $(A.3)$ now follows immediately from the facts that

$$\lim_{n \downarrow 0} f(n) = \alpha, \qquad (A.20)$$

$$\alpha < \frac{\alpha + \beta}{2} \Leftrightarrow \alpha < \beta \qquad (A.21)$$

and from the concavity of the function $f(n)$ iff $\alpha < \beta$ (see Lemma 1). $\square$

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